

PROBABILISTIC LOCAL AND GLOBAL WELL-POSEDNESS FOR THE NONLINEAR WAVE EQUATION ON $B_2 \times \mathbb{T}$

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ABSTRACT. We establish probabilistic local and global well-posedness results for the nonlinear wave equation, posed on the domain $B_2 \times \mathbb{T}$, with randomly chosen initial data having radial symmetry in the B_2 variable, and with vanishing Dirichlet boundary conditions on $\partial B_2 \times \mathbb{T}$.

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1. INTRODUCTION

In the present work, we consider probabilistic local and global well-posedness results for the cubic nonlinear wave equation on the periodic cylinder $B_2 \times \mathbb{T}$, where $B_2 \subset \mathbb{R}^2$ denotes the planar unit ball $\{|x'| < 1\}$ and with radially imposed on the B_2 variable.

We begin with some preliminary notation. Write $x = (x', x_3) \in B_2 \times \mathbb{T}$ and let (e_n) be the sequence of radial eigenfunctions of $-\Delta$ on B_2 (with vanishing Dirichlet boundary conditions), ordered so that the associated eigenvalues λ_n^2 are in increasing order. Introducing the notation

$$z_{n,n'} := \sqrt{\lambda_n^2 + (n')^2}. \quad (1)$$

for the eigenvalues of the operator $-\Delta$ on $B_2 \times \mathbb{T}$ (restricted to functions which are radial in the x' variable), we observe that the space of eigenfunctions for this operator is generated by the family of maps

$$(x', x_3) \mapsto e_n(x') e^{2\pi i n' x_3}, \quad n \in \mathbb{N}, n' \in \mathbb{Z}.$$

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We now state the initial value problem studied in this paper. In particular, letting (Ω, \mathcal{M}, p) be a given probability space, we shall establish local and global well-posedness results for the nonlinear wave equation with defocusing nonlinearity,

$$(NLW) \quad \begin{cases} u_{tt} - \Delta u + |u|^\gamma u = 0, & \text{on } I \times (B_2 \times \mathbb{T}), \\ (u, u_t)(0) = (\phi_\omega, \psi_\omega), & \text{on } B_2 \times \mathbb{T}, \\ u(t, x_3)|_{\partial B_2} = 0, & t \in I, x_3 \in \mathbb{T}, \end{cases}$$

with random initial data $(\phi_\omega, \psi_\omega)$ given by

$$\phi_\omega = \sum_{n \geq 1} \sum_{n' \in \mathbb{Z}} \frac{h_{n,n'}(\omega)}{(z_{n,n'})^\alpha} e_n(x_1, x_2) e^{2\pi i n' x_3}, \quad (2)$$

and

$$\psi_\omega = \sum_{n \geq 1} \sum_{n' \in \mathbb{Z}} \frac{k_{n,n'}(\omega)}{(z_{n,n'})^{\alpha-1}} e_n(x_1, x_2) e^{2\pi i n' x_3}, \quad (3)$$

for $\omega \in \Omega$, where $\alpha > 1$ is fixed and (h_n) and (k_n) are taken as sequences of real-valued Gaussian random variables on Ω .

Let $H_x^s(B_2 \times \mathbb{T})$, $s \in \mathbb{R}$, denote the space of functions $f : B_2 \times \mathbb{T} \rightarrow \mathbb{C}$ with

$$f(x) = \sum_{n \geq 1} \sum_{n' \in \mathbb{Z}} a_{n,n'} e_n(x_1, x_2) e^{2\pi i n' x_3},$$

for some $(a_{n,n'}) \in \ell^2(n, n')$ such that the associated norm

$$\|f\|_{H_x^s} := \left(\sum_{n \geq 1} \sum_{n' \in \mathbb{Z}} \langle z_{n,n'} \rangle^{2s} |a_{n,n'}|^2 \right)^{1/2} \quad (4)$$

is finite. Then, for each given $\alpha > 1$, the initial data $(\phi_\omega, \psi_\omega)$ belongs almost surely to each of the spaces

$$H_x^s(B_2 \times \mathbb{T}) \times H_x^{s-1}(B_2 \times \mathbb{T})$$

with $s < \alpha - 1$.

Before stating our main results, we give a brief overview of some related works in the literature. For an overview of the classical deterministic well-posedness theory for NLW, see, for instance, [21]. In the probabilistic setting, study of local and global well-posedness properties of the nonlinear wave and nonlinear Schrödinger (NLS) equations, as well as related equations of dispersive type, began with the foundational works [1, 2, 3] of Bourgain treating NLS on the one, two, and three dimensional torus (see also the work of Lebowitz, Rose and Speer [16]).

Following these works, a large number of authors have studied related issues. While for a comprehensive survey of such results to date, we refer to the survey article [9], we point out several particular works which have relevance for our present study. In [14], Colliander and Oh studied NLS on \mathbb{T} at low regularity, and made use of a probabilistic formulation of the high/low truncation method of Bourgain, obtaining a class of global well-posedness results below L^2 ; see also [17]. In [18], Nahmod, Oh, Rey-Bellet and Staffilani studied derivative NLS on \mathbb{T} , making use of a suitable gauge transformation, while in [20] Nahmod and Staffilani have recently studied probabilistic local well-posedness for the energy-critical problem on \mathbb{T}^3 . For NLW on the flat 3D torus \mathbb{T}^3 , Burq and Tzvetkov have obtained probabilistic global

well-posedness results [13], using a variety of energy-based considerations. See also [19] for recent results concerning the Navier-Stokes system.

Turning to the case of more general compact manifolds, Burq and Tzvetkov have recently established a probabilistic local well-posedness theory for cubic NLW on arbitrary compact three-dimensional manifolds, and moreover have obtained almost sure global well-posedness for NLS and NLW [22, 23, 11, 12] on the 2D and 3D unit balls B_2 and B_3 , making use of the invariance of the Gibbs measure. In a recent series of works, the author and J. Bourgain have improved on these results, obtaining almost sure global well-posedness for Gibbs measure evolutions for NLW on the 3D unit ball with arbitrary energy-subcritical nonlinearity in [8], for NLS on the 2D unit ball with higher power nonlinearities in [6], and for cubic NLS on the 3D unit ball in [7]; see also [5] for an overview of these results.

In the present work, we view the domain $B_2 \times \mathbb{T}$ as a case which interpolates between the B_3 and \mathbb{T}^3 settings – at a technical level, this interpolation results from the difference in eigenvalue asymptotics between $B_2 \times \mathbb{T}$ and B_3 , along with the presence of L_x^∞ eigenfunction estimates in the periodic variable.

We now state our first main result, which gives local well-posedness for the initial value problem (NLW). In establishing this result, we will use the following reformulated form of the equation, which arises from the substitution $u = w + i(\sqrt{-\Delta})^{-1}w_t$:

$$\begin{cases} iu_t - \sqrt{-\Delta}u + (\sqrt{-\Delta})^{-1}(|\operatorname{Re} u|^\gamma \operatorname{Re} u) &= 0, \\ u|_{t=t_0} &= \phi_\omega. \end{cases} \quad (5)$$

with

$$\phi_\omega = \sum_{n \geq 1} \sum_{n' \in \mathbb{Z}} \frac{g_{n,n'}(\omega)}{(z_{n,n'})^\alpha} e_n(x_1, x_2) e^{2\pi i n' x_3}, \quad (6)$$

and where $(g_{n,n'})$ is a sequence of IID complex-valued Gaussian random variables on (Ω, \mathcal{M}, p) .

Denoting the evolution operator associated to the linear equation $iu_t - (\sqrt{-\Delta})u = 0$ by

$$S(t)\phi = \sum_{n,n'} a_n e_n(x_1, x_2) e^{2\pi i(n'x_3 + z_{n,n'}t)} \quad (7)$$

whenever $\phi : B_2 \times \mathbb{T} \rightarrow \mathbb{C}$ is given by

$$\phi(x) = \sum_{n,n'} a_n e_n(x_1, x_2) e^{2\pi i n' x_3}, \quad (8)$$

we shall consider solutions to (5) in the sense of the Duhamel formula

$$u(t) = S(t)u(0) - i \int_0^t S(t-t')(\sqrt{-\Delta})^{-1} [|\operatorname{Re} u|^\gamma \operatorname{Re} u(t')] dt'. \quad (9)$$

With this notation in hand, our local well-posedness result takes the form of the following theorem:

Theorem 1.1 (Local well-posedness for (NLW)). *Fix $\alpha > 1$, and let $\gamma \in \mathbb{R}$ be given satisfying*

$$1 \leq \gamma < \min \left\{ \frac{6\alpha - 1}{3(3 - 2\alpha)}, \frac{4\alpha - 3 + \sqrt{4\alpha^2 - 35\alpha + 54}}{3(3 - 2\alpha)} \right\} \quad (10)$$

if $1 < \alpha < 3/2$, and $1 \leq \gamma < \infty$ if $\alpha \geq 3/2$.

Then for every $A \geq 1$ and $t_0 \in \mathbb{R}$ there exists a set $\Sigma_A \subset \Omega$ with

$$p(\Sigma_A) \leq C_1 \exp(-C_2 A^2)$$

such that for all $\omega \in \Omega \setminus \Sigma_A$ the initial value problem (5) has a unique solution u on the interval $I = [t_0, t_0 + A^{-c}]$, with

$$u - S(t - t_0)\phi_\omega \in X^{s,b}(I) \quad (11)$$

for some $s \in [\frac{1}{2}, 1]$ (depending on γ) and $b > \frac{1}{2}$.

Recall that initial data of the form (2)–(3) is, almost surely in ω , in the regularity class $H_x^{\alpha-1-\epsilon} \times H_x^{\alpha-2-\epsilon}$, $\epsilon > 0$. When α and γ satisfy $\alpha \leq \frac{5}{2} - \frac{2}{\gamma}$ the problem therefore belongs to the ill-posed (i.e. supercritical) regime, and probabilistic considerations are essential.

The proof of Theorem 1.1 is based on a fixed-point analysis in $X^{s,b}$ spaces; see Section 2 for the definition of these spaces in our context. We outline three key ingredients in the argument, each of which makes essential use of the product structure of the domain:

- a class of local-in-time Strichartz estimates adapted to our $B_2 \times \mathbb{T}$ setting, making use of eigenfunction expansions of the type given in (8); see Proposition 2.2 — it is important to note that the present setting is made delicate by the presence of “essentially repeating” eigenvalues (that is, $(n_1, n'_1) \neq (n_2, n'_2)$ with $z_{n_1, n'_1} \sim z_{n_2, n'_2}$) — to account for this, we make use of the technique of spectral projectors (this is described further in Section 2.2 below; see also [10]),
- a class of large-deviation type estimates for the linear evolution, adapted to the product structure of the domain and making full use of the eigenvalue asymptotics in our setting; see Section 3, and
- an estimate of the nonlinearity which allows us to control the $X^{s,b}$ norm of the nonlinear term in the Duhamel formula by a suitable $L_x^p L_t^2$ norm; see Lemma 4.1.

For $\alpha > 1$ sufficiently close to 1, the restrictions on the nonlinearity given by (10) in Theorem 1.1 can be generalized; in particular, applying Case 3 of Appendix A shows that a larger range of γ can be treated, provided that the regularity s appearing in (11) is allowed to be in the range $0 < s \leq 1$.

We now turn to global well-posedness, for which the main component of the argument is contained in the proof of the following theorem:

Theorem 1.2. *Let $\alpha > 11/7$ be given and suppose that $1 \leq \gamma < 4$ satisfies*

$$(2 - \alpha) \left(2\gamma + \frac{2(\gamma + 1)}{\gamma + 2} - 2 \right) - \alpha + \frac{3}{2} - \frac{1}{\gamma + 1} < 0. \quad (12)$$

Then for each $\epsilon > 0$ and $T > 0$, there exists a set $\Sigma_\epsilon \subset \Omega$ with

$$p(\Sigma_\epsilon) \leq \epsilon$$

and such that for every $\omega \in \Omega \setminus \Sigma_\epsilon$ the initial value problem (NLW) has a unique solution $u \in C([0, T]; H_x^1(B_2 \times \mathbb{T}))$.

The proof of Theorem 1.2 is based on a suitable form of the local well-posedness theory given by Theorem 1.1, combined with an application of Bourgain's high-low Fourier truncation method for low-regularity global well-posedness, and is composed of three main steps:

- a deterministic local well-posedness result for the equation (5), which allows us to construct solutions on small time intervals chosen according to the H_x^1 norm of the initial data, with control over the energy; see Lemma 6.1,
- a probabilistic local well-posedness result for a perturbed form of (5), the proof of which is closely related to the proof of Theorem 1.1; see Lemma 6.2 — note that this result gives H_x^1 regularity for the nonlinear part of the evolution which is suitable for iteration with the lemma of the first step, and
- a sequence of iterative applications of the two local well-posedness results described above, which allows for solutions to be constructed up to the given time T ; an important part of the argument at this stage is to keep track of the increase in H_x^1 norm of the data at each stage of the iteration.

We refer the reader to Section 6 for further discussion of this argument as well as relevant references, including prior applications of the high/low method in related probabilistic settings.

Theorem 1.2 can be informally described as an *almost* global well-posedness result for initial data outside a singular set of small measure. It is by now well established in the literature that elementary probabilistic considerations imply that statements of this form lead immediately to almost sure global well-posedness results. In particular, we obtain the following corollary.

Corollary 1.3 (Global well-posedness for (NLW)). *Let $\alpha > 11/7$ be given and suppose that $1 \leq \gamma < 4$ satisfies the condition (12) of Theorem 1.2. Then there exists a set $\Sigma_* \subset \Omega$ with*

$$p(\Sigma_*) = 0$$

and such that for every $\omega \in \Omega \setminus \Sigma_*$ the initial value problem (NLW) has a unique solution $u \in C([0, \infty); H_x^1(B_2 \times \mathbb{T}))$.

Proof. We argue as in the proof of [14, Theorem 2]. For each $n \in \mathbb{N}$, let $\Sigma_n \subset \Omega$ be the set given by Theorem 1.2 with $\epsilon = 2^{-n}$ and $T = n$. Now, set

$$\Sigma_* := \bigcap_{N \geq 1} \bigcup_{n \geq N} \Sigma_n.$$

We then have

$$p(\Sigma_*) \leq \inf_{N \geq 1} \sum_{n=N}^{\infty} p(\Sigma_n) \lesssim \inf_{N \geq 1} 2^{-N} = 0$$

while for all $\omega \in \Omega \setminus \Sigma_*$ there exists $N \geq 1$ such that we have $\omega \in \Omega \setminus \Sigma_n$ for all $n \geq N$, and thus the corresponding initial value problem (NLW) is globally well-posed. \square

We remark that the result of Theorem 1.2 includes nonlinearities for which the corresponding deterministic problem with equivalent regularity initial data is ill-posed. As we noted above, this corresponds to values of α and $1 \leq \gamma < 4$ satisfying (12) with $\alpha < \frac{5}{2} - \frac{2}{\gamma}$. Numerical computations show that such values of α exist for γ in the range $3.33187 \leq \gamma < 4$.

We conclude this introduction by giving an outline of the rest of this paper. In Section 2, we establish some notational conventions which will be useful in the rest of the article, including the specification of the function spaces of $X^{s,b}$ type which will be used for our local-wellposedness arguments. We also obtain the relevant Strichartz estimates. In Section 3 we obtain the relevant probabilistic estimates, which are bounds of large deviation type, and which rely heavily on the product structure of the domain $B_2 \times \mathbb{T}$ in order to obtain the optimal amount of integrability. Section 4 is then devoted to estimates of the nonlinearity, which are applied in Section 5 to obtain the proof of Theorem 1.1. The proof of Theorem 1.2 is then contained in Section 6, using the high-low method of Bourgain.

2. PRELIMINARIES AND FUNCTION SPACES

In this section, we establish some notation and specify the main function spaces (of $X^{s,b}$ type) which will underlie our arguments in the remainder of the paper.

We will frequently write $\langle x \rangle = (1 + |x|^2)^{1/2}$, and make use of the notation $z_{n,n'}$ defined in (1) for $n \in \mathbb{N}$ and $n' \in \mathbb{Z}$. In this context, summations over n will typically be taken over \mathbb{N} , while summations over n' will be taken over \mathbb{Z} . As usual, we will interchangeably use the notations $A \lesssim B$ and $A \leq CB$, $C > 0$, and the value of C may change from line to line (unless otherwise indicated). Moreover, $I \subset \mathbb{R}$ will be used to denote an arbitrary time interval with $|I| \leq 1$.

We will also use the notation μ_α to denote the probability measure induced by the Gaussian process

$$\omega \mapsto \phi_\omega = \sum_{n \geq 1} \sum_{n' \in \mathbb{Z}} \frac{g_{n,n'}(\omega)}{(z_{n,n'})^\alpha} e_n(x_1, x_2) e^{2\pi i n' x_3},$$

i.e. the measure given by

$$\mu_\alpha(A) = p(\{\omega : \phi_\omega \in A\}).$$

We now recall some basic properties of the eigenfunctions and eigenvalues of $-\Delta$ on B_2 which will play a fundamental role in our analysis below.¹ Recall that (e_n) and (λ_n^2) denote the sequences of radial eigenfunctions and associated eigenvalues of $-\Delta$ on B_2 (with vanishing boundary conditions, and arranged so that $\lambda_n < \lambda_{n+1}$

¹It is illuminating to compare (13)–(14) with the B_3 estimates used in [8, 12]. The difference in homogeneity, combined with the presence of L_x^∞ estimates on the eigenfunctions $e^{2\pi i n' x_3}$ in the \mathbb{T} variable (such estimates are indeed classically relevant; see, e.g. [9]), in some sense corresponds to the intuition that our results interpolate between the B_3 and \mathbb{T}^3 cases.

for all n). It follows from standard estimates on Bessel functions that one has the asymptotic bound

$$\left| \lambda_n - \left(n\pi - \frac{\pi}{4} \right) \right| \leq Cn^{-1}. \quad (13)$$

Moreover, similar arguments yield the following L^p estimates for the eigenfunctions e_n :

$$\|e_n\|_{L_x^p(B_2)} \lesssim \begin{cases} 1, & 1 \leq p < 4, \\ \log(2+n)^{1/4}, & p = 4, \\ n^{\frac{1}{2} - \frac{2}{p}}, & p > 4. \end{cases} \quad (14)$$

2.1. Function spaces. We now specify the function spaces which will be used in the rest of the paper. For each $s \in \mathbb{R}$, we shall use $H_x^s(B_2 \times \mathbb{T})$ to denote the usual Sobolev space given by the norm (4). As described in the introduction, we shall also make use of $X^{s,b}$ spaces adapted to our context. In particular, fixing $s \in \mathbb{R}$, $b > \frac{1}{2}$, and a time interval $I \subset \mathbb{R}$, the space $X^{s,b}(I)$ will denote the space of all functions having representations

$$f(t, x) = \sum_{m, n, n'} f_{m, n, n'} e_n(x_1, x_2) e^{2\pi i(n'x_3 + mt)}. \quad (15)$$

such that the norm

$$\|f\|_{X^{s,b}(I)} := \inf_{(f_{m, n, n'}) \subset \mathbb{C}} \left(\sum_{m, n, n'} \langle m - z_{n, n'} \rangle^{2b} \langle z_{n, n'} \rangle^{2s} |f_{m, n, n'}|^2 \right)^{1/2}$$

is finite, where the infimum is taken over all sequences $(f_{m, n, n'})$ leading to the representation (15) on $I \times (B_2 \times \mathbb{T})$.

With this notation, we note that for all $s > 0$, $b > \frac{1}{2}$, one has the (continuous) embedding

$$X^{s,b} \hookrightarrow C_t(I; H_x^s). \quad (16)$$

An additional embedding property is established in Section 4 as Lemma 4.2.

Moreover, we will also make use of the following estimate for the $X^{s,b}$ norm of the linear evolution $S(t)f$ (where $f \in H_x^s$ is arbitrary) given by (7):

Lemma 2.1. *Fix $s > 0$, $b > \frac{1}{2}$, $0 < T < 1/2$, and let $\chi \in C_c^\infty(\mathbb{R})$ be such that $\text{supp } \chi \subset [0, T/2]$. Then there exists a constant $C > 0$ such that the inequality*

$$\|\chi(t)[S(t)f]\|_{X^{s,b}([0,T])} \leq C\|f\|_{H_x^s}$$

holds for all $f \in H_x^s(B_2 \times \mathbb{T})$.

Proof. We write

$$S(t)f = \sum_{n, n'} a_{n, n'} e_n(x_1, x_2) e^{2\pi i(n'x_3 + z_{n, n'}t)}$$

for some sequence $(a_{n, n'}) \in \ell^2$ with $\sum \langle z_{n, n'} \rangle^{2s} |a_{n, n'}|^2 < \infty$. Setting,

$$b_{n, n', m} := \int_0^1 \chi(t) e^{2\pi i(z_{n, n'} - m)t} dt,$$

we therefore obtain

$$\chi(t)S(t)f = \sum_{n,n',m} a_{n,n'} b_{n,n',m} e_n(x_1, x_2) e^{2\pi i(n'x_3 + mt)}.$$

Taking the $X^{s,b}([0, T])$ norm, we obtain

$$\begin{aligned} \|\chi(t)S(t)f\|_{X^{s,b}([0,T])} &\leq \left(\sum_{n,n',m} \langle m - z_{n,n'} \rangle^{2b} \langle z_{n,n'} \rangle^{2s} |a_{n,n'}|^2 |b_{n,n',m}|^2 \right)^{1/2} \\ &\leq \left(\sup_{n,n'} \sum_m \langle m - z_{n,n'} \rangle^{2b} |b_{n,n',m}|^2 \right)^{1/2} \left(\sum_{n,n'} \langle z_{n,n'} \rangle^{2s} |a_{n,n'}|^2 \right)^{1/2} \end{aligned}$$

Now, using integration by parts to obtain the bound $|b_{n,n',m}| \lesssim |m - z_{n,n'}|^{-2}$, it follows that

$$\|\chi(t)S(t)f\|_{X^{s,b}} \lesssim \|f\|_{H_x^s}$$

as desired. \square

2.2. Strichartz estimates. We conclude this section by establishing a suitable form of the linear Strichartz estimates associated with the operator $S(t)$ defined in (7).

Proposition 2.2. *For every p, q , and $s > 0$ satisfying $2 \leq p, q < \infty$ and*

$$s > \frac{1}{2} - \frac{1}{q} + \max \left\{ \frac{1}{2} - \frac{2}{p}, 0 \right\},$$

one has

$$\|S(t)f\|_{L_x^p L_t^q(I)} \lesssim \|f\|_{H_x^s}$$

for every $f \in H_x^s(B_2 \times \mathbb{T})$ and $I \subset \mathbb{R}$ with $|I| \leq 1$.

In order to proceed with the proof of Proposition 2.2, we will need to introduce a spectral projection operator. In particular, for

$$f(x) = \sum_{n,n'} a_{n,n'} e_n(x_1, x_2) e^{2\pi i n' x_3},$$

let $Af : B_2 \times \mathbb{T} \rightarrow \mathbb{C}$ be defined by

$$Af(x) := \sum_{n,n'} a_{n,n'} \lfloor z_{n,n'} \rfloor e_n(x_1, x_2) e^{2\pi i n' x_3}, \quad x \in B_2 \times \mathbb{T},$$

and note that the operator $A : f \mapsto Af$ is then bounded on H_x^s for any $s \geq 0$.

We will also make use of the associated evolution $S_A(t)f$ for $t > 0$ given by

$$(S_A(t)f)(x) := \sum_{n,n'} a_{n,n'} e_n(x_1, x_2) e^{2\pi i(n'x_3 + \lfloor z_{n,n'} \rfloor t)}.$$

To establish Proposition 2.2, we note that after setting $u = S(t)f$, the equality

$$iu_t + Au = (A - \sqrt{-\Delta})u$$

can be rewritten in integral form as

$$S(t)f = S_A(t)f + i \int_0^t S_A(t-t')[(A - \sqrt{-\Delta})S(t')f]dt'. \quad (17)$$

It therefore suffices to establish the following lemma (see, e.g. the last step in the proof of Theorem 2 in [10], as well as the references cited there):

Lemma 2.3. *For p, q and s be as stated in Proposition 2.2, one has*

$$\|S_A(t)f\|_{L_x^p L_t^q(I)} \lesssim \|f\|_{H_x^s}$$

for every $f \in H_x^s(B_2 \times \mathbb{T})$ and $I \subset \mathbb{R}$ with $|I| \leq 1$.

Proof. Writing

$$f(x) = \sum_{n,n'} a_{n,n'} e_n(x_1, x_2) e^{2\pi i n' x_3}$$

and using the Sobolev inequality in time, we obtain

$$\begin{aligned} \|S_A(t)f\|_{L_x^p L_t^q} &\lesssim \left\| \sum_{n,n'} a_{n,n'} e_n(x_1, x_2) e^{2\pi i(n' x_3 + \lfloor z_{n,n'} \rfloor t)} \right\|_{L_x^p H_t^{\frac{1}{2} - \frac{1}{q}}} \\ &= \left\| \left(\sum_k \langle k \rangle^{1 - \frac{2}{q}} \left| \sum_{\substack{n,n' \\ \lfloor z_{n,n'} \rfloor = k}} a_{n,n'} e_n(x_1, x_2) e^{2\pi i n' x_3} \right|^2 \right)^{1/2} \right\|_{L_x^p} \\ &\leq \left\| \left(\sum_k \langle k \rangle^{1 - \frac{2}{q}} N(k) \left(\sum_{\substack{n,n' \\ \lfloor z_{n,n'} \rfloor = k}} |a_{n,n'}|^2 |e_n(x_1, x_2)|^2 \right) \right)^{1/2} \right\|_{L_x^p} \quad (18) \end{aligned}$$

where we have set

$$N(k) := \#\{(n, n') : \lfloor z_{n,n'} \rfloor = k\}.$$

It now follows from (13) and standard arithmetic estimates counting the number of lattice points on circles (see for instance [1, 2] as well as [6, Section 2.2] and the references cited in these works) that for every $\epsilon > 0$ one has the bound

$$N(k) \leq C(\epsilon) k^\epsilon \text{ for all } k \geq 1$$

for a suitable constant $C(\epsilon) > 0$ (in fact, one has the stronger estimate $N(k) \lesssim \exp(c \log k / \log \log(k))$ for some $c > 0$, but we will not need this refinement in our setting).

Fixing $\epsilon > 0$, we therefore obtain

$$\begin{aligned} (18) &\lesssim \left\| \left(\sum_k \langle k \rangle^{1 - \frac{2}{q} + \epsilon} \left(\sum_{\substack{n,n' \\ \lfloor z_{n,n'} \rfloor = k}} |a_{n,n'}|^2 |e_n(x_1, x_2)|^2 \right) \right)^{1/2} \right\|_{L_x^p} \\ &\leq \left(\sum_k \sum_{\substack{n,n' \\ \lfloor z_{n,n'} \rfloor = k}} \langle k \rangle^{1 - \frac{2}{q} + \epsilon} |a_{n,n'}|^2 \|e_n(x_1, x_2)\|_{L_x^p}^2 \right)^{1/2}. \quad (19) \end{aligned}$$

To conclude the proof of the lemma, we observe that for $p \leq 4$, (19) is bounded by

$$\left(\sum_k \sum_{\substack{n, n' \\ \lfloor z_{n, n'} \rfloor = k}} \langle k \rangle^{1 - \frac{2}{q} + 2\epsilon} |a_{n, n'}|^2 \right)^{1/2} \lesssim \|f\|_{H_x^{\frac{1}{2} - \frac{1}{q} + \epsilon}} \quad (20)$$

while for $p > 4$, one gets the bound

$$\left(\sum_k \sum_{\substack{n, n' \\ \lfloor z_{n, n'} \rfloor = k}} \langle k \rangle^{2 - \frac{2}{q} - \frac{4}{p} + \epsilon} |a_{n, n'}|^2 \right)^{1/2} \lesssim \|f\|_{H_x^{1 - \frac{1}{q} - \frac{2}{p} + \frac{\epsilon}{2}}}. \quad (21)$$

Combining (19) with (20)–(21) completes the proof of Lemma 2.3. \square

3. PROBABILISTIC ESTIMATES OF THE LINEAR EVOLUTION

We now establish a collection of probabilistic estimates on the linear evolution. These estimates (along with Lemma 6.3 of Section 6.1 below, which is a variant that takes into account frequency localization) comprise the main probabilistic component of our argument.

For $\alpha > 1$ and $0 \leq s < \alpha - 1$, define

$$\rho_*(\alpha, s) = \begin{cases} \frac{4}{3 - 2(\alpha - s)}, & \text{if } 1 < \alpha - s < \frac{3}{2}, \\ \infty, & \text{if } \alpha - s \geq \frac{3}{2}. \end{cases} \quad (22)$$

and

$$\rho_*(\alpha) = \rho_*(\alpha, 0) \quad (23)$$

Our probabilistic estimates now take the form of the following lemma:

Lemma 3.1. *Fix $T > 0$ and $\alpha > 1$. Then there exists $c > 0$ such that the estimates:*

(i) *for all $0 < s < \alpha - 1$ and $1 \leq p < \rho_*(\alpha, s)$,*

$$\mu_\alpha(\{\phi : \|(\sqrt{-\Delta})^s \phi\|_{L^p} > \lambda\}) \lesssim e^{-c\lambda^2}, \quad (24)$$

and

(ii) *for all $(p, q) \in [1, \infty)^2$ with $1 \leq p < \rho_*(\alpha)$ and $2 \leq q < \infty$,*

$$\mu_\alpha(\{\phi : \|S(t)\phi\|_{L_x^p L_t^q} > \lambda\}) \lesssim e^{-c\lambda^2}, \quad (25)$$

are valid for all $\lambda > 0$, where ϕ is randomly chosen initial data of the form (6).

It is important to note that the estimates of Lemma 3.1 make essential use of the product structure of the domain $B_2 \times \mathbb{T}$. Indeed, the reader may find it useful to compare the range $p < \rho_*(\alpha)$ with the corresponding range in the B_3 case [8]. In our setting, the gain in integrability is essentially a consequence of the B_2 eigenfunction bounds (14) and the fact that eigenfunctions on \mathbb{T} (and indeed \mathbb{T}^d for $d \geq 1$) are bounded in L_x^∞ .

Proof of Lemma 3.1. Note that for each of the statements (i) and (ii) there is no loss of generality in considering the case $p > 4$.

Indeed, for $p \leq 4$, $\lambda > 0$ and $\epsilon > 0$, it follows from Hölder's inequality that there exists $c > 0$ with

$$\begin{aligned} \mu_\alpha(\{\phi : \|(\sqrt{-\Delta})^s \phi\|_{L_x^p(B_2 \times \mathbb{T})} > \lambda\}) \\ \leq \mu_\alpha(\{\phi : \|(\sqrt{-\Delta})^s \phi\|_{L_x^{4+\epsilon}} > c\lambda\}) \end{aligned}$$

and

$$\begin{aligned} \mu_\alpha(\{\phi : \|S(t)\phi\|_{L_x^p L_t^q} > \lambda\}) \\ \leq \mu_\alpha(\{\phi : \|S(t)\phi\|_{L_x^{4+\epsilon} L_t^q} > c\lambda\}). \end{aligned}$$

We therefore let $p > 4$ be given, for which (14) gives

$$\|e_n\|_{L_x^p(B_2)} \lesssim n^{\frac{1}{2} - \frac{2}{p}}.$$

To establish (24), suppose that $p < \rho_*(\alpha, s)$, and let $\lambda > 0$ be given. Now, fix $r \geq p$ and write

$$\begin{aligned} \mu_\alpha(\{\phi : \|(\sqrt{-\Delta})^s \phi\|_{L^p} > \lambda\}) &\leq \frac{1}{\lambda^r} \mathbb{E} \left[\|(\sqrt{-\Delta})^s \phi\|_{L^p}^r \right] \\ &\leq \frac{1}{\lambda^r} \left\| \left(\mathbb{E} \left[|(\sqrt{-\Delta})^s \phi(x)|^r \right] \right)^{1/r} \right\|_{L^p}^r \\ &\lesssim \frac{(\sqrt{r})^r}{\lambda^r} \left\| \left(\sum_{n,n'} \frac{|e_n(x_1, x_2)|^2}{(z_{n,n'})^{2(\alpha-s)}} \right)^{1/2} \right\|_{L^p}^r. \end{aligned}$$

This quantity is then equal to

$$\begin{aligned} &\frac{(\sqrt{r})^r}{\lambda^r} \left\| \sum_{n,n'} (z_{n,n'})^{2(s-\alpha)} |e_n(x_1, x_2)|^2 \right\|_{L^{p/2}}^{r/2} \\ &\leq \frac{(\sqrt{r})^r}{\lambda^r} \left(\sum_{n,n'} (z_{n,n'})^{2(s-\alpha)} \|e_n(x_1, x_2)\|_{L^p}^2 \right)^{r/2} \\ &\lesssim \frac{(\sqrt{r})^r}{\lambda^r} \left(\sum_{n,n'} (z_{n,n'})^{2(s-\alpha)} n^{2(\frac{1}{2} - \frac{2}{p})} \right)^{r/2} \end{aligned} \quad (26)$$

where to obtain the second inequality we have used (14) and recalled that $p > 4$ holds by assumption.

Now, fixing an auxiliary parameter $\beta \in (0, 1)$, recalling the hypothesis $s < \alpha - 1$, and using the eigenvalue bound $\lambda_n \gtrsim n$ (for n sufficiently large), we have

$$\sum_{n,n'} (z_{n,n'})^{2(s-\alpha)} n^{2(\frac{1}{2} - \frac{2}{p})} \lesssim \sum_{n' \in \mathbb{Z}} \left(\frac{1}{(n')^{2(\alpha-s)\beta}} \sum_{n \geq 1} \frac{1}{n^{2(\alpha-s)(1-\beta) - 1 + \frac{4}{p}}} \right) \quad (27)$$

We now examine two cases. Suppose first that $1 < \alpha - s < \frac{3}{2}$, and thus $\rho_*(\alpha, s) < \infty$. We then have $p < \frac{4}{3-2\alpha+2s}$, which implies that we may choose $\beta \in (0, 1)$ such that

$$2(\alpha - s)\beta > 1 \quad (28)$$

and

$$2(\alpha - s)(1 - \beta) - 1 + \frac{4}{p} > 1. \quad (29)$$

Let β be such a value. Conditions (28) and (29) then imply that the right side of (27) is finite. In view of (26), we have thus obtained

$$\mu_\alpha(\{\phi : \|(\sqrt{-\Delta})^s \phi\|_{L^p} > \lambda\}) \lesssim (\sqrt{r}/\lambda)^r. \quad (30)$$

Minimizing the right hand side of (30) over all $r \geq p$, this completes the proof of (24) in the case $\alpha - s < \frac{3}{2}$.

Alternatively, suppose that $\alpha - s \geq \frac{3}{2}$ (recall that in this case p is subject only to the restriction $p > 4$). For any $\beta \in (\frac{1}{3}, \frac{1}{3} + \frac{4}{3p})$, one then obtains

$$2(\alpha - s)\beta \geq 3\beta > 1$$

and

$$2(\alpha - s)(1 - \beta) - 1 + \frac{4}{p} \geq 3(1 - \beta) - 1 + \frac{4}{p} > 1.$$

It follows that the right side of (27) is again finite; as a consequence, we once more obtain the bound (30), so that minimizing in the choice of r completes the proof of (24) in the case $\alpha - s \geq \frac{3}{2}$.

To complete the proof of the lemma, it remains to show (25). For this, we argue as above. In particular, letting $4 < p < \rho_*(\alpha)$, $2 \leq q < \infty$ and $\lambda > 0$ be given, we have

$$\mu_\alpha(\{\phi : \|S(t)\phi\|_{L_x^p L_t^q} > \lambda\}) \leq \frac{(\sqrt{r})^r}{\lambda^r} \left\| \left(\sum_{n,n'} \frac{|e_n(x_1, x_2)|^2}{(z_{n,n'})^{2\alpha}} \right)^{1/2} \right\|_{L_x^p L_t^q}^r \quad (31)$$

for every $r \geq \max\{p, q\}$. Arguing as in (26), we then obtain the bound

$$(31) \leq \frac{(\sqrt{r})^r}{\lambda^r} \left(\sum_{n,n'} (z_{n,n'})^{-2\alpha} n^{2(\frac{1}{2} - \frac{2}{p})} \right)^{r/2}. \quad (32)$$

Now, fixing $\beta \in (0, 1)$, we bound the right side of (32) by a multiple of

$$\sum_{n' \in \mathbb{Z}} \left(\frac{1}{\langle n' \rangle^{2\alpha\beta}} \sum_{n \geq 1} \frac{1}{n^{2\alpha(1-\beta) - 1 + \frac{4}{p}}} \right).$$

To complete the proof, we again split into two cases. Suppose first that $1 < \alpha < \frac{3}{2}$. In this case, $p < \rho_*(\alpha)$ becomes $p < \frac{4}{3-2\alpha}$, and thus there exists $\beta \in (0, 1)$ satisfying

$$2\alpha\beta > 1 \quad (33)$$

and

$$2\alpha(1 - \beta) - 1 + \frac{4}{p} > 1. \quad (34)$$

We therefore obtain

$$\mu_\alpha(\{\phi : \|S(t)\phi\|_{L_x^p L_t^q} > \lambda\}) \lesssim (\sqrt{r}/\lambda)^r.$$

Optimizing in the choice of r now gives (25) as desired.

Alternatively, suppose that $\alpha \geq \frac{3}{2}$. In this case, for all $4 < p < \infty$ we may choose any $\beta \in (\frac{1}{3}, \frac{1}{3} + \frac{4}{3p})$ and obtain (33)–(34), so that the argument concludes as before. This completes the proof of the lemma. \square

4. ESTIMATES OF THE NONLINEARITY

In this section we establish two lemmas which will provide estimates for the nonlinear term of the Duhamel formula (9). These lemmas, when combined with the probabilistic bounds of Section 3, will facilitate the proof of the local well-posedness result stated in Theorem 1.2 (as well as the subsequent global results) by allowing us to close a contraction mapping argument in suitable $X^{s,b}$ norms.

We begin with the following lemma.

Lemma 4.1. *Fix $0 < s \leq 1$ and suppose that $p \in [1, \infty)$ satisfies*

$$\begin{aligned} (i) \quad & p > \frac{2}{2-s} \text{ if } 0 < s < \frac{1}{2}, \quad (ii) \quad p > \frac{6}{5-2s} \text{ if } \frac{1}{2} \leq s < 1, \text{ and} \\ (iii) \quad & p \geq 2 \text{ if } s = 1. \end{aligned}$$

Then there exists $\epsilon > 0$ such that for every $b \in (\frac{1}{2}, \frac{1}{2} + \epsilon)$ there exists a constant $C_b > 0$ such that, for all $f \in L_x^p(B_2 \times \mathbb{T}; L_t^2(I))$,

$$\left\| \int_0^t S(t-\tau)(\sqrt{-\Delta})^{-1} f(\tau) d\tau \right\|_{X^{s,b}(I)} \leq C \|f\|_{L_x^p(B_2 \times \mathbb{T}; L_t^2(I))}.$$

Proof. Fix $\epsilon > 0$ (to be chosen later in the argument) and let $b \in \mathbb{R}$ be given such that $\frac{1}{2} < b < \frac{1}{2} + \epsilon$. Now, writing

$$\begin{aligned} & \int_0^t S(t-t')(\sqrt{-\Delta})^{-1} f(t') dt' \\ &= \int_0^t S(t-t')(\sqrt{-\Delta})^{-1} \left(\sum_{m,n,n'} \hat{f}(m,n,n') e_n(x_1, x_2) e^{2\pi i(n'x_3 + mt')} \right) dt' \\ &= \sum_{m,n,n'} \frac{\hat{f}(m,n,n')}{z_{n,n'}} e_n(x_1, x_2) e^{2\pi i(n'x_3 + z_{n,n'}t)} \int_0^t e^{2\pi i(m - z_{n,n'})t'} dt' \\ &= \sum_{m,n,n'} \frac{\hat{f}(m,n,n')}{z_{n,n'}(m - z_{n,n'})} e_n(x_1, x_2) e^{2\pi i n' x_3} (e^{2\pi i m t} - e^{2\pi i z_{n,n'} t}) \end{aligned}$$

we have

$$\left\| \int_0^t S(t-t')(\sqrt{-\Delta})^{-1} f(t') dt' \right\|_{X^{s,b}} \lesssim (I) + (II)$$

with

$$(I) := \left(\sum_{m,n,n'} \frac{|\hat{f}(m,n,n')|^2}{\langle z_{n,n'} \rangle^{2(1-s)} \langle m - z_{n,n'} \rangle^{2(1-b)}} \right)^{1/2}$$

and

$$(II) := \left(\sum_{\tilde{m}, n, n'} |a_{\tilde{m}, n, n'}|^2 \langle \tilde{m} - z_{n, n'} \rangle^{2b} \langle z_{n, n'} \rangle^{2s} \right)^{1/2},$$

$$a_{\tilde{m}, n, n'} := \sum_m \frac{\hat{f}(m, n, n')}{z_{n, n'}(m - z_{n, n'})} \int_0^1 e^{2\pi i z_{n, n'} t} e^{2\pi i \tilde{m} t} dt.$$

An application of the Cauchy-Schwartz inequality gives the bound $(II) \lesssim (I)$. We estimate (I) by duality, writing

$$(I) \lesssim \left(\sum_{m, n, n'} \frac{|\hat{f}(m, n, n')|^2}{\langle z_{n, n'} \rangle^{2(1-s)} \langle m - z_{n, n'} \rangle^{2(1-b)}} \right)^{1/2}$$

$$= \sup_{g \in \mathcal{G}} \int_I \int_{B_2 \times \mathbb{T}} f(t, x) \overline{g(t, x)} dx dt$$

where the set \mathcal{G} consists of functions $g : I \times (B_2 \times \mathbb{T}) \rightarrow \mathbb{C}$ of the form

$$g(t, x) = \sum_{m, n, n'} \frac{g_{m, n, n'}}{\langle z_{n, n'} \rangle^{1-s} \langle m - z_{n, n'} \rangle^{1-b}} e_n(x_1, x_2) e^{2\pi i (n' x_3 + m t)}$$

which satisfy

$$\sum_{m, n, n'} |g_{m, n, n'}|^2 \leq 1.$$

For each $g \in \mathcal{G}$, note that by the Hölder inequality, one has

$$\int_I \int_{B_2 \times \mathbb{T}} f \bar{g} \leq \|f\|_{L_x^p L_t^2} \|g\|_{L_x^q L_t^2}$$

with $\frac{1}{p} + \frac{1}{q} = 1$. We now consider cases depending on the value of s . Suppose first that $0 < s < \frac{1}{2}$ holds. We then get the inequalities

$$\begin{aligned} \|g\|_{L_x^q L_t^2} &\leq \left\| \left(\sum_m \left(\sum_{n, n'} \frac{|g_{m, n, n'}|}{\langle z_{n, n'} \rangle^{1-s} \langle m - z_{n, n'} \rangle^{1-b}} |e_n(x_1, x_2)| \right) \right)^2 \right\|_{L_x^q}^{1/2} \\ &\leq \left\| \sum_m \left(\sum_{n, n'} |g_{m, n, n'}|^2 \right) \left(\sum_{n, n'} \frac{|e_n(x_1, x_2)|^2}{\langle z_{n, n'} \rangle^{2(1-s)} \langle m - z_{n, n'} \rangle^{2(1-b)}} \right) \right\|_{L_x^{q/2}}^{1/2} \\ &= \left\| \sum_{n, n'} \frac{|e_n(x_1, x_2)|^2}{\langle z_{n, n'} \rangle^{1-2(s+\epsilon)}} \beta_{n, n'} \right\|_{L_x^{q/2}}^{1/2} \end{aligned} \quad (35)$$

with

$$\beta_{n, n'} := \sum_m \left(\sum_{n, n'} |g_{m, n, n'}|^2 \right) \langle z_{n, n'} \rangle^{-1-2\epsilon} \langle m - z_{n, n'} \rangle^{-2(1-b)}.$$

It now follows that we have the bound

$$(35) \lesssim \left(\sum_{n, n'} \frac{\|e_n(x_1, x_2)\|_{L_x^q}^2}{\langle z_{n, n'} \rangle^{1-2(s+\epsilon)}} \beta_{n, n'} \right)^{1/2}$$

$$\begin{aligned}
&\lesssim \left(\sup_{n,n'} \frac{\|e_n(x_1, x_2)\|_{L_x^q}}{\langle z_{n,n'} \rangle^{\frac{1}{2} - (s+\epsilon)}} \right) \left(\sum_{n,n'} \beta_{n,n'} \right)^{1/2} \\
&\lesssim \sup_{n,n'} \frac{\|e_n(x_1, x_2)\|_{L_x^q}}{\langle z_{n,n'} \rangle^{\frac{1}{2} - (s+\epsilon)}}.
\end{aligned}$$

In view of the eigenfunction estimates (14), this quantity is finite for all $q < \frac{2}{s}$ (this corresponds to the restriction $p > \frac{2}{2-s}$). This completes the proof when $0 < s < \frac{1}{2}$.

Suppose now that $\frac{1}{2} < s < 1$. By Minkowski's inequality for integrals and the Sobolev embedding, we then get (for $q \geq 2$)

$$\|g\|_{L_x^q L_t^2} \leq \|g\|_{L_t^2 L_x^q} \lesssim \|g\|_{L_t^2 H_x^{\frac{3}{2} - \frac{3}{q}}}$$

which is bounded by $\|g\|_{X^{1-s, 1-b}(I)} \leq 1$ when $\frac{3}{2} - \frac{3}{q} < 1 - s$ (which corresponds to $p > \frac{6}{5-2s}$).

It remains to consider the case $s = 1$, where the desired conclusion follows by taking $q = 2$ (and thus $p = 2$) to obtain

$$\|g\|_{L_{t,x}^2} \leq \|g\|_{X^{1-s, 1-b}(I)} \leq 1.$$

This completes the proof of Lemma 4.1. \square

The second lemma of this section expresses an embedding of the form

$$X^{s,b} \hookrightarrow L_x^p L_t^q$$

for suitable values of $s > 0$ and $2 \leq p, q < \infty$. As remarked in the introduction, our arguments in the $B_2 \times \mathbb{T}$ setting of the present work are somewhat delicate, since we must appeal to the Strichartz estimates of Proposition 2.2.

Lemma 4.2. *Fix $s > 0$, $b > \frac{1}{2}$, and $2 \leq p, q < \infty$. Suppose further that $s > \frac{1}{2} - \frac{1}{q} + \max\{\frac{1}{2} - \frac{2}{p}, 0\}$. Then there exists $C > 0$ and $\delta > 0$ such that*

$$\|f\|_{L_x^p L_t^q} \leq C \|f\|_{X^{s,b}} \quad (36)$$

for all $f \in X^{s,b}(I)$.

Proof. Suppose $f \in X^{s,b}$ satisfies $\|f\|_{X^{s,b}} \leq 1$. We may then write

$$\|f\|_{L_x^p L_t^q} = \left\| \sum_{m,n,n'} \frac{a_{m,n,n'}}{\langle z_{n,n'} \rangle^s \langle m - z_{n,n'} \rangle^b} e_n(x_1, x_2) e^{2\pi i(n'x_3 + mt)} \right\|_{L_x^p L_t^q} \quad (37)$$

where $(a_{m,n,n'})$ is a sequence of complex numbers with $\sum_{m,n,n'} |a_{m,n,n'}|^2 \leq 1$. Making the change of variable $\ell = m - z_{n,n'}$ in the summation, we obtain

$$\begin{aligned}
(37) &= \left\| \sum_{\ell} \frac{1}{\langle \ell \rangle^b} \sum_{n,n'} \frac{a_{z_{n,n'} + \ell, n, n'}}{\langle z_{n,n'} \rangle^s} e_n(x_1, x_2) e^{2\pi i(n'x_3 + z_{n,n'}t)} e^{2\pi i\ell t} \right\|_{L_x^p L_t^q} \\
&= \left\| \sum_{\ell} \frac{b_{\ell}}{\langle \ell \rangle^b} \sum_{n,n'} \frac{c_{\ell, n, n'}}{\langle z_{n,n'} \rangle^s} e_n(x_1, x_2) e^{2\pi i(n'x_3 + z_{n,n'}t)} e^{2\pi i\ell t} \right\|_{L_x^p L_t^q}
\end{aligned}$$

where we have set

$$b_\ell := \left(\sum_{n,n'} |a_{z_{n,n'}+\ell,n,n'}|^2 \right)^{1/2}, \quad c_{\ell,n,n'} := \frac{a_{z_{n,n'}+\ell,n,n'}}{b_\ell}$$

and where the summation in ℓ is taken over the (countable) index set

$$\mathcal{J} = \left\{ m - z_{n,n'} : m, n, n' \in \mathbb{Z}, n \geq 1 \right\} \subset \mathbb{R}.$$

with the convention that, for $r \in \mathbb{R}$, one sets $a_{r,n,n'} = 0$ when r is not an integer.

Invoking Minkowski's inequality, we obtain

$$\begin{aligned} (37) &\leq \sum_{\ell \in \mathcal{J}} \frac{b_\ell}{\langle \ell \rangle^b} \left\| \sum_{n,n'} \frac{c_{\ell,n,n'}}{\langle z_{n,n'} \rangle^s} e_n(x_1, x_2) e^{2\pi i(n'x_3 + z_{n,n'}t)} \right\|_{L_x^p L_t^q} \\ &\lesssim \sup_{\ell} \left\| \sum_{n,n'} \frac{c_{\ell,n,n'}}{\langle z_{n,n'} \rangle^s} e_n(x_1, x_2) e^{2\pi i(n'x_3 + z_{n,n'}t)} \right\|_{L_x^p L_t^q} \end{aligned} \quad (38)$$

where to obtain the last inequality we have observed that, in view of the Cauchy-Schwartz inequality and the conditions $b > \frac{1}{2}$, $\sum_{m,n,n'} |a_{m,n,n'}|^2 \leq 1$, one has the bound

$$\sum_{\ell} \frac{b_\ell}{\langle \ell \rangle^b} \leq \left(\sum_{\ell} \frac{1}{\langle \ell \rangle^{2b}} \right)^{1/2} \left(\sum_{\ell} b_\ell^2 \right)^{1/2} \lesssim 1.$$

To estimate the quantity in (38), fix $\ell \in \mathcal{J}$, $\delta > 0$, and use Proposition 2.2 to obtain the bound

$$\begin{aligned} &\left\| \sum_{n,n'} \frac{c_{\ell,n,n'}}{\langle z_{n,n'} \rangle^s} e_n(x_1, x_2) e^{2\pi i(n'x_3 + z_{n,n'}t)} \right\|_{L_x^p L_t^q} \\ &\lesssim \left\| \sum_{n,n'} \frac{c_{\ell,n,n'}}{\langle z_{n,n'} \rangle^s} e_n(x_1, x_2) e^{2\pi i n' x_3} \right\|_{H_x^\sigma} \end{aligned} \quad (39)$$

for all $\sigma \in \mathbb{R}$ with $\sigma > \frac{1}{2} - \frac{1}{q} + \max\{\frac{1}{2} - \frac{2}{p}, 0\}$.

The right side of (39) is equal to

$$\left(\sum_{n,n'} \frac{|c_{\ell,n,n'}|^2}{\langle z_{n,n'} \rangle^{2(s-\sigma)}} \right)^{1/2},$$

so that choosing $\sigma > 0$ such that

$$\frac{1}{2} - \frac{1}{q} + \max\left\{\frac{1}{2} - \frac{2}{p}, 0\right\} < \sigma < s$$

we obtain

$$\|f\|_{L_x^p L_t^q} \leq C \sup_{\ell \in \mathcal{J}} \left(\sum_{n,n'} |c_{\ell,n,n'}|^2 \right)^{1/2}.$$

The desired inequality (36) now follows by observing that, for each $\ell \in \mathcal{J}$, the inequality $\sum_{n,n'} |c_{\ell,n,n'}|^2 \leq 1$ holds by construction. \square

5. LOCAL WELL-POSEDNESS: CONTRACTION IN $X^{s,b}$ SPACES

In this section we give the proof of Theorem 1.2, the local well-posedness result. As we described in the introduction, the proof is based on a contraction-mapping argument in the spaces $X^{s,b}$, using the results of Sections 2, 3 and 4. To simplify notation, in this section and in the remainder of the paper we shall make use of the abbreviation

$$F(z) = |\operatorname{Re}(z)|^\gamma \operatorname{Re}(z), \quad z \in \mathbb{C}, \quad (40)$$

when convenient.

Proof of Theorem 1.1. Fix $0 < s \leq 1$ and $b > \frac{1}{2}$. In view of the time translation symmetry, we may take $t_0 = 0$ without any loss of generality. We now proceed by a fixed point argument. In particular, for each $A \geq 1$ we shall identify positive constants T and R (each depending on A) and construct a set Σ_A satisfying the conditions stated above such that for all $\omega \in \Omega \setminus \Sigma_A$ the map $\Phi = \Phi_\omega : B_R \rightarrow B_R$ (where $B_R := \{v \in X^{s,b}([0, T]) : \|v\|_{X^{s,b}} \leq R\}$) given by

$$[\Phi_\omega(v)](t, x) := \int_0^t S(t-t')(\sqrt{-\Delta})^{-1} \left[|\operatorname{Re}(S(t')\phi_\omega + v(t'))|^\gamma \operatorname{Re}(S(t')\phi_\omega + v(t')) \right] dt'$$

is a contraction.

Let $A \geq 1$ be given. Fix $R > 0$ and $0 < T < 1$ to be determined later in the argument, and set $I = [0, T]$. Let $v \in X^{s,b}(I)$ be given with $\|v\|_{X^{s,b}} \leq R$. Now, fix $p \geq 1$ and $\epsilon > 0$ satisfying the conditions of Lemma 4.1. Invoking that lemma, we get

$$\begin{aligned} \|\Phi(v)\|_{X^{s,b}(I)} &\lesssim \|S(t)\phi + v(t)\|_{L_x^{p(\gamma+1)}(B_2 \times \mathbb{T}; L_t^{2(\gamma+1)}(I))}^{\gamma+1} \\ &\lesssim \|S(t)\phi\|_{L_x^{p(\gamma+1)}(B_2 \times \mathbb{T}; L_t^{2(\gamma+1)})}^{\gamma+1} \\ &\quad + \|v\|_{L_x^{p(\gamma+1)}(B_2 \times \mathbb{T}; L_t^{2(\gamma+1)})}^{\gamma+1} \end{aligned} \quad (41)$$

under the condition $b < \frac{1}{2} + \epsilon$.

To estimate the linear evolution $S(t)\phi$, we appeal to the probabilistic considerations of Section 3. In particular, if p is chosen such that

$$2 \leq p(\gamma+1) < \rho_*(\alpha) \quad (42)$$

(where $\rho_*(\alpha)$ is as in (23)), then an application of Lemma 3.1 gives the bound

$$\|S(t)\phi\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}} \leq A \quad (43)$$

for all $\omega \in \Omega \setminus \Sigma_A$, with

$$\Sigma_A := \{\omega \in \Omega : \|S(t)\phi_\omega\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}} > A\},$$

$$\mu(\Sigma_A) \lesssim \exp(-cA^2).$$

On the other hand, to estimate the $L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}$ norm of v , we note that Lemma 4.2 gives

$$\|v\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}} \leq CT^{1/\epsilon} \|v\|_{X^{s,b}}, \quad (44)$$

for some $\epsilon > 0$ (see (47) below), provided that the parameters s and p satisfy the condition

$$s > \frac{1}{2} - \frac{1}{2(\gamma+1)} + \max \left\{ \frac{1}{2} - \frac{2}{p(\gamma+1)}, 0 \right\}. \quad (45)$$

Indeed, assuming (45), the bound (44) is obtained by choosing $q > 2(\gamma+1)$ such that

$$s > \frac{1}{2} - \frac{1}{q} + \max \left\{ \frac{1}{2} - \frac{2}{p(\gamma+1)}, 0 \right\}, \quad (46)$$

and using the Hölder inequality followed by Lemma 4.2 with the norm $L_x^{p(\gamma+1)} L_t^q$:

$$\begin{aligned} \|v\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}} &\lesssim T^{\frac{1}{2(\gamma+1)} - \frac{1}{q}} \|v\|_{L_x^{p(\gamma+1)} L_t^q} \\ &\lesssim T^{\frac{1}{2(\gamma+1)} - \frac{1}{q}} \|v\|_{X^{s,b}}, \end{aligned}$$

Taking q as large as possible while satisfying (46), we obtain the desired estimate (44) for all $\epsilon > 0$ satisfying

$$\epsilon > \left(\frac{1}{2(\gamma+1)} - \left[\frac{1}{2} - s + \max \left\{ \frac{1}{2} - \frac{2}{p(\gamma+1)}, 0 \right\} \right]_+ \right)^{-1}. \quad (47)$$

where the notation $[t]_+ = \max\{t, 0\}$ is used to denote the positive part of $t \in \mathbb{R}$.

Combining (41) with (43) and (44), we have

$$\|\Phi_\omega(v)\|_{X^{s,b}(I)} \leq C_1 A^{\gamma+1} + C_2 (T^{1/\epsilon} R)^{\gamma+1}.$$

for all $\omega \in \Omega \setminus \Sigma_A$, provided that there exist $s > 0$ and $p \geq 1$ satisfying the condition of Lemma 3.1 together with (42) and (45). Note that the condition (10) implies that such values of s and p exist (see Appendix A for further comments on this point). Choosing $R = 2 \max\{C_1, C_2\} A^{\gamma+1}$ and

$$T < \left(\frac{1}{2C_2 R^\gamma} \right)^{\frac{\epsilon}{\gamma+1}}$$

now gives

$$\|\Phi_\omega(v)\|_{X^{s,b}(I)} \leq R$$

for all $\omega \in \Omega \setminus \Sigma_A$. This shows that for all such ω , the map Φ_ω carries the set B_R to itself.

To conclude the desired existence and uniqueness result, it remains to show that Φ_ω is a contraction on B_R whenever ω is not in Σ_A . This follows almost immediately from the above arguments and the elementary inequality

$$|F(a) - F(b)| \lesssim |a - b|(|a|^\gamma + |b|^\gamma). \quad (48)$$

We include the details for the convenience of the reader: let $v, w \in B_R$ be given and suppose that s and p are as above. Arguing as before, we obtain

$$\begin{aligned} \|\Phi_\omega(v) - \Phi_\omega(w)\|_{X^{s,b}} &\lesssim \|F(S(t)\phi + v(t)) - F(S(t)\phi + w(t))\|_{L_x^p L_t^2} \\ &\lesssim \| |v(t) - w(t)| (|S(t)\phi + v(t)|^\gamma + |S(t)\phi + w(t)|^\gamma) \|_{L_x^p L_t^2} \\ &\leq \|v - w\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}} \\ &\quad \cdot \| |S(t)\phi + v(t)|^\gamma + |S(t)\phi + w(t)|^\gamma \|_{L_x^{\frac{p(\gamma+1)}{\gamma}} L_t^{\frac{2(\gamma+1)}{\gamma}}}, \end{aligned} \quad (49)$$

where the third line follows from (48) and the fourth line results from an application of Hölder's inequality. Then, using Hölder's inequality (in time) and Lemma 4.2 to estimate the norm of $v - w$, we get

$$\begin{aligned}
(49) &\lesssim T^{1/\epsilon} \|v - w\|_{X^{s,b}} \\
&\quad \cdot (\|S(t)\phi\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}^\gamma + \|v\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}^\gamma + \|w\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}^\gamma) \\
&\lesssim T^{(\gamma+1)/\epsilon} \|v - w\|_{X^{s,b}} (\|S(t)\phi\|_{L_x^{p(\gamma+1)} L_t^q}^\gamma + \|v\|_{X^{s,b}}^\gamma + \|w\|_{X^{s,b}}^\gamma)
\end{aligned}$$

with $q > 2(\gamma + 1)$ chosen as before (see (46) above).

Recalling that v and w each belong to B_R , this gives

$$\|\Phi_\omega(v) - \Phi_\omega(w)\|_{X^{s,b}} \leq CT^{(\gamma+1)/\epsilon} (A^\gamma + 2R^\gamma) \|v - w\|_{X^{s,b}}$$

for all $\omega \in \Omega \setminus \Sigma_A$. Recalling the choice of R above as a multiple of a power of A , and choosing

$$T < \min \left\{ \left(\frac{1}{2C_2 R^\gamma} \right)^{\frac{\epsilon}{\gamma+1}}, \left(\frac{1}{2C(A^\gamma + 2R^\gamma)} \right)^{\frac{\epsilon}{\gamma+1}} \right\},$$

we obtain the desired contraction property for all such ω .

It follows immediately that there is a unique $v \in B_R$ such that $\Phi(v) = v$. We conclude the proof of the proposition by noting that the function $u(t) = S(t)\phi_\omega + v(t)$ is the desired solution of the initial value problem. \square

Remark 5.1. Under the hypotheses of Theorem 1.1, the constructed solution satisfies the bound

$$\|u\|_{L_t^\infty(I; H_x^{s'})} \leq CA^{\gamma+2}, \quad 0 < s' < \min\{s, \alpha - 1\}.$$

with s as in the statement of the theorem. Indeed, this is an immediate consequence of (11), when combined with the embedding $X^{s,b} \hookrightarrow C_t(I; H_x^s)$ (see (16) in Section 2) and the conservation of H^s norms for the linear propagator. \diamond

6. GLOBAL WELL-POSEDNESS: HIGH/LOW TRUNCATION

We now turn our attention to global well-posedness for (NLW), for which we will make use of the high-low method of Bourgain (introduced in [4], and used in the probabilistic setting in [14]; see also [15] and the recent work [17]).

The argument is based on iterative applications of local well-posedness results for the initial-value problems

$$\begin{cases} iv_t - \sqrt{-\Delta}v + (\sqrt{-\Delta})^{-1}F(v) &= 0, \\ v|_{t=t_0} &= \phi_1. \end{cases} \quad (50)$$

and

$$\begin{cases} iw_t - \sqrt{-\Delta}w + (\sqrt{-\Delta})^{-1}[F(v+w) - F(v)] &= 0, \\ w|_{t=t_0} &= \phi_2. \end{cases} \quad (51)$$

where ϕ_1 and ϕ_2 are initial data to be specified at each iteration, and where F is given as in (40) (that is, $F(z) = |\operatorname{Re}(z)|^\gamma \operatorname{Re}(z)$ for $z \in \mathbb{C}$).

We begin with an analysis of the local well-posedness theory for these initial value problems. Before proceeding to this analysis (which is the topic of the next subsection), we shall find it useful to recall the *energy* associated to (50). In particular, for each $\varphi \in H^1(B_2 \times \mathbb{T})$, one defines

$$E(\varphi) := \frac{1}{2} \int_{B_2 \times \mathbb{T}} |(\sqrt{-\Delta})\varphi|^2 dx + \frac{1}{\gamma + 2} \int_{B_2 \times \mathbb{T}} |\operatorname{Re} \varphi|^{\gamma+2} dx. \quad (52)$$

Then, for every solution $u : I \times B_2 \times \mathbb{T}$ of (50), one has the identity

$$E(u(t)) = E(u(0))$$

for all $t \in I$.²

6.1. Local well-posedness for the initial value problems (50) and (51). In this subsection, we establish two variants of the local well-posedness result obtained in Theorem 1.1 of Section 5, adapted to the initial value problems (50) and (51).

The first such result is contained in the following lemma, which applies to finite-energy initial data ϕ_1 .

Lemma 6.1 (Deterministic local well-posedness for (50)). *Fix $1 \leq \gamma < 4$ and let $A \geq 1$, $b > \frac{1}{2}$ be given. Then there exist constants $c_1, c_2, C > 0$ such that for every $\phi_1 \in H_x^1$ with*

$$\|\phi_1\|_{H_x^1} \leq A \quad (53)$$

the initial value problem (50) has a unique solution v on the interval $I = [t_0, t_0 + c_1 A^{-c_2})$ with

$$\|v\|_{X^{1,b}(I)} \leq CA.$$

Moreover, the constructed solution v satisfies the bound

$$\sup_{t \in I} E(v(t)) \leq CA^{\gamma+2}. \quad (54)$$

Proof. We assume without loss of generality that $t_0 = 0$. Let $\phi_1 \in H_x^1(B_2 \times \mathbb{T})$ be given with $\|\phi_1\|_{H_x^1} \leq A$, and fix $R, T > 0$ to be determined later in the argument.

To establish the existence and uniqueness claims, we set $I = [0, T]$ and invoke a contraction mapping argument as in the proof of Theorem 1.1. In particular, for each $v \in X^{1,b}([0, T])$ with $\|v\|_{X^{1,b}} \leq R$, we define

$$[\Phi(v)](t) = S(t)\phi_1 + i \int_{k\delta}^t S(t-t')(\sqrt{-\Delta})^{-1} F(v(t')) dt'.$$

As usual, it will suffice to show that Φ is a map from $B_R := \{v \in X^{1,b}(I) : \|v\|_{X^{1,b}} \leq R\}$ to itself; the contraction property on B_R then follows from similar arguments and the inequality (48). Fixing $p \geq 2$ (which will be specified further

²Indeed, one can see the conserved nature of this quantity in several ways. Perhaps the simplest observation which leads to this is to recall the Hamiltonian structure $iu_t = (\sqrt{-\Delta})^{-1} \frac{\partial E}{\partial \bar{u}}$ of (5), with $E = E(u)$ as above. Alternatively, as noted in [12] (where the calculation is applied to the equation satisfied by the finite-dimensional projection) one can multiply (5) by $\Delta \bar{u} - |\operatorname{Re}(u)|^\gamma \operatorname{Re} u$, integrate in x and use standard integration by parts formulas.

momentarily), we then have, by the Minkowski inequality, Lemma 2.1, and Lemma 4.1,

$$\begin{aligned}\|\Phi(v)\|_{X^{1,b}} &\leq \|S(t)\phi_1\|_{X^{1,b}} + \left\| \int_0^t S(t-t')(\sqrt{-\Delta})^{-1}F(v(t'))dt' \right\|_{X^{1,b}} \\ &\lesssim \|\phi_1\|_{H_x^1} + \|v\|_{L_x^{p(\gamma+1)}L_t^{2(\gamma+1)}}^{\gamma+1}.\end{aligned}$$

In view of Lemma 4.2, since p moreover satisfies the condition

$$1 > \frac{1}{2} - \frac{1}{2(\gamma+1)} + \max \left\{ \frac{1}{2} - \frac{2}{p(\gamma+1)}, 0 \right\},$$

we see that arguing as in the justification of (44) in the proof of Theorem 1.1 gives the bound

$$\|\Phi(v)\|_{X^{1,b}} \lesssim \|\phi_1\|_{H_x^1} + (T^{1/\epsilon}\|v\|_{X^{1,b}})^{\gamma+1}.$$

Now, recalling (53) and $v \in B_R$, we get

$$\|\Phi(v)\|_{X^{1,b}} \leq CA + C(T^{1/\epsilon}R)^{\gamma+1}.$$

Setting $R = 2CA$ and letting T satisfy

$$T < (AR^{-(\gamma+1)})^{\frac{\epsilon}{\gamma+1}} = \frac{1}{(2C)^\epsilon} A^{-\frac{\gamma\epsilon}{\gamma+1}},$$

we obtain that Φ is a map from B_R to itself.

It remains to show the energy bound (54). To see this, we observe that by the conservation of energy, the assumption (53), and the Sobolev inequality, one obtains, for all $t \in I$,

$$E(v(t)) = E(v(0)) \lesssim A^2 + \|\phi_1\|_{H_x^{\frac{3\gamma}{2(\gamma+2)}}}^{\gamma+2}. \quad (55)$$

The condition $\gamma < 4$ now implies $\frac{3\gamma}{2(\gamma+2)} < 1$, so that (in view of (53) and $A \geq 1$) the right side of (55) is bounded by a constant multiple of $A^{\gamma+2}$. This establishes (54) as desired. \square

We now turn to the relevant local well-posedness result for (51). In the context of the iterative procedure performed in the next section, we will take initial data of the form

$$\phi_2(x) = \sum_{\{n,n': z_{n,n'} \geq L\}} \frac{g_{n,n'}(\omega)}{(z_{n,n'})^\alpha} e_n(x_1, x_2) e^{2\pi i n' x_3}, \quad x \in B_2 \times \mathbb{T}, \quad (56)$$

which (almost surely in ω) belongs to the spaces $H_x^{\alpha-1-\epsilon}$ for all $\epsilon > 0$.

Lemma 6.2 (Probabilistic local well-posedness for (51)). *Let $\alpha > 1$ be given and suppose that γ satisfies*

$$1 \leq \gamma < \min \left\{ \frac{2\alpha-1}{3-2\alpha}, 4 \right\} \text{ if } \alpha < \frac{3}{2}, \quad \text{and} \quad 1 \leq \gamma < 4 \text{ if } \alpha \geq \frac{3}{2}.$$

Then for every $A, B \geq 1$ and $t_0 \in \mathbb{R}$ there exist a set $\Sigma_A \subset \Omega$ and a time interval $I = [t_0, t_0 + T]$ with

$$p(\Sigma_A) \leq C \exp(-cA^{c'}) \text{ and } 0 < T < c(A+B)^{-c'}$$

such that for every $\omega \in \Omega \setminus \Sigma_A$ and all $v : I \times B_2 \times \mathbb{T} \rightarrow \mathbb{C}$ satisfying, for some $b > \frac{1}{2}$ sufficiently close to $\frac{1}{2}$,

$$\|v\|_{X^{1,b}(I)} \leq B, \quad (57)$$

the initial value problem (51) with initial data (56) has a unique solution w on the interval I with

$$w - S(t - t_0)\phi_2 \in X^{1,b}(I).$$

Before giving the proof of Lemma 6.2, it will be useful for us to establish a slightly refined version of the probabilistic bounds obtained in Lemma 3.1 for the linear propagator $S(t)$. In these refined estimates, we take into account the high-frequency structure of the data (56). We also use this opportunity to specify the dependence on the length T of the time interval $[t_0, t_0 + T]$.

Lemma 6.3 (High-frequency probabilistic estimate). *Fix $\alpha > 1$ and $1 \leq p < \rho_*(\alpha)$, $2 \leq q < \infty$. Then there exist constants $c, C > 0$ such that if $\tau > 0$ satisfies $\frac{3}{2} - \frac{2}{p} < \tau < \alpha$, then the inequality*

$$\mu_\alpha(\{\phi_2 : \|S(t)\phi_2\|_{L_x^p L_t^q([t_0, t_0+T])} > \lambda T^{1/q} L^{-(\alpha-\tau)}\}) \leq C e^{-c\lambda^2} \quad (58)$$

holds for every $t_0 \in \mathbb{R}$, $T > 0$, $L \in \mathbb{N}$, and $\lambda > 0$, where each ϕ_2 is taken of the form (56).

Proof. We argue as in the proof of Lemma 3.1, letting α and p satisfy $\alpha < \frac{3}{2}$ and $p > 4$ without loss of generality. Let $\lambda > 0$ be given. The quantity on the left side of (58) is then bounded by a multiple of

$$\begin{aligned} & \left(\frac{L^{\alpha-\tau} \sqrt{r}}{\lambda} \right)^r \left\| \left(\sum_{n,n'} \frac{|e_n(x_1, x_2)|^2}{(z_{n,n'})^{2\alpha}} \right)^{1/2} \right\|_{L_x^p}^r \\ & \leq \left(\frac{\sqrt{r}}{\lambda} \right)^r \left\| \left(\sum_{n,n'} \frac{|e_n(x_1, x_2)|^2}{(z_{n,n'})^{2\tau}} \right)^{1/2} \right\|_{L_x^p}^r \end{aligned} \quad (59)$$

for every $r \geq \max\{p, q\}$. Arguing as in (26), we then get

$$(59) \leq \frac{(\sqrt{r})^r}{\lambda^r} \left(\sum_{n,n'} (z_{n,n'})^{-2\tau} n^{2(\frac{1}{2} - \frac{2}{p})} \right)^{r/2}. \quad (60)$$

so that after fixing $\beta \in (0, 1)$ we obtain the bound

$$\sum_{n' \in \mathbb{Z}} \left(\frac{1}{\langle n' \rangle^{2\tau\beta}} \sum_{n \geq 1} \frac{1}{n^{2\tau(1-\beta) - 1 + \frac{4}{p}}} \right).$$

The condition on τ now implies that there exists $\beta \in (0, 1)$ satisfying

$$2\tau\beta > 1 \quad \text{and} \quad 2\tau(1-\beta) - 1 + \frac{4}{p} > 1,$$

and the desired result follows by optimizing the choice of r . \square

We now return to the proof of Lemma 6.2.

Proof of Lemma 6.2. We again proceed by a fixed point argument, recalling that there is no loss of generality in fixing $t_0 = 0$. In particular, fix $s = 1$ and let

$$p \geq 2$$

be given, fix $I = [0, T]$ with $T > 0$ to be determined at the conclusion of the argument, and suppose that $v : I \times B_2 \times \mathbb{T} \rightarrow \mathbb{C}$ satisfies (57).

Now, for each $\omega \in \Omega$ and $\tilde{w} \in X^{s,b}([0, T])$, define $\Phi(\tilde{w}) : I \times B_2 \times \mathbb{T} \rightarrow \mathbb{C}$ by

$$[\Phi(\tilde{w})](t, x) := \int_0^t S(t-t')(\sqrt{-\Delta})^{-1} \left[F(S(t')\phi_{\omega,2}^{(k+1)} + v(t') + \tilde{w}(t')) - F(v(t')) \right] dt'$$

for $(t, x) \in I \times B_2 \times \mathbb{T}$ (note that $\Phi(\tilde{w})$ is chosen to ensure that if \tilde{w} is a fixed point of the map $\tilde{w} \mapsto \Phi(\tilde{w})$ then $w(t) = S(t)\phi_2 + \tilde{w}(t)$ solves (51)).

Fix $R > 0$ and $b > \frac{1}{2}$ to be specified later. We now show that the map $\tilde{w} \mapsto \Phi(\tilde{w})$ maps the ball

$$B_R := \{v \in X^{s,b}([0, T]) : \|v\|_{X^{s,b}} \leq R\}$$

to itself (for suitable choices of $\omega \in \Omega$ and the parameters T , R and b). Let $\tilde{w} \in B_R$ be given. Then, applying Lemma 4.1 as in the proof of Theorem 1.1 we obtain the bound

$$\|\Phi(\tilde{w})\|_{X^{s,b}(I)} \lesssim \|F(S(t')\phi_{\omega,2}^{(k+1)} + \tilde{w}(t') + v(t')) - F(v(t'))\|_{L_x^p L_t^2} \quad (61)$$

provided that b is chosen sufficiently close to $\frac{1}{2}$.

Now, using the elementary inequality (48) we obtain

$$\begin{aligned} (61) &\lesssim \| |S(t)\phi_2 + \tilde{w}(t)| (|S(t)\phi_2 + \tilde{w}(t)|^\gamma + |v(t)|^\gamma) \|_{L_x^p L_t^2} \\ &\leq \|S(t)\phi_2 + \tilde{w}(t)\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}} \\ &\quad \cdot \| |S(t)\phi_2 + \tilde{w}(t)|^\gamma + |v(t)|^\gamma \|_{L_x^{\frac{p(\gamma+1)}{\gamma}} L_t^{\frac{2(\gamma+1)}{\gamma}}}, \end{aligned} \quad (62)$$

where we have used the Hölder inequality to obtain the last line.

It now follows from Minkowski's inequality that (62) is bounded by a multiple of

$$\begin{aligned} &\|S(t)\phi_2 + \tilde{w}(t)\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}^{\gamma+1} \\ &\quad + \|S(t)\phi_2 + \tilde{w}(t)\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}} \|v(t)\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}^\gamma \\ &\lesssim \|S(t)\phi_2\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}^{\gamma+1} + \|\tilde{w}(t)\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}^{\gamma+1} \\ &\quad + (\|S(t)\phi_2\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}} + \|\tilde{w}(t)\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}) \|v(t)\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}^\gamma. \end{aligned} \quad (63)$$

To estimate $\|S(t)\phi_2\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}$, we fix $\tau > 0$ with $\frac{3}{2} - \frac{2}{p(\gamma+1)} < \tau < \alpha$, and restrict ω to the set

$$\Omega \setminus \Sigma_A$$

with

$$\Sigma_A := \{\omega : \|S(t)\phi_2\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}} > AT^{\frac{1}{2(\gamma+1)}} L^{-(\alpha-\tau)}\},$$

where we have recalled that, by assumption, the function $\phi_2 = \phi_2(\omega) : B_2 \times \mathbb{T} \rightarrow \mathbb{C}$ is of the form (56).

This restriction on ω gives the bound

$$(63) \lesssim A^{\gamma+1} T^{\frac{1}{2}} L^{-(\alpha-\tau)(\gamma+1)} + \|\tilde{w}(t)\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}^{\gamma+1} \\ + (AT^{\frac{1}{2(\gamma+1)}} L^{-(\alpha-\tau)} + \|\tilde{w}(t)\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}) \|v(t)\|_{L_x^{p(\gamma+1)} L_t^{2(\gamma+1)}}^{\gamma}, \quad (64)$$

while Lemma 3.1 ensures that we have the estimate

$$p(\Sigma_A) \lesssim \exp(-cA^2)$$

provided that p is chosen to satisfy the condition (42) appearing in the proof of Theorem 1.1.

We now use the Hölder inequality and Lemma 4.2 to estimate the remaining norms appearing in (64). In particular, assuming s satisfies the condition (45) appearing in the proof of Theorem 1.1, we choose $q > 2(\gamma + 1)$ satisfying (46). Now, using Hölder in time gives

$$(64) \leq A^{\gamma+1} T^{\frac{1}{2}} L^{-(\alpha-\tau)(\gamma+1)} + T^{\frac{q-2(\gamma+1)}{2q}} \|\tilde{w}(t)\|_{L_x^{p(\gamma+1)} L_t^q}^{\gamma+1} \\ + (AT^{\frac{1}{2(\gamma+1)}} L^{-(\alpha-\tau)} + T^{\frac{q-2(\gamma+1)}{2q(\gamma+1)}} \|\tilde{w}(t)\|_{L_x^{p(\gamma+1)} L_t^q}) (T^{\frac{q-2(\gamma+1)}{2q(\gamma+1)}} \|v(t)\|_{L_x^{p(\gamma+1)} L_t^q})^{\gamma}.$$

so that by Lemma 4.2 we get

$$(64) \leq A^{\gamma+1} T^{\frac{1}{2}} L^{-(\alpha-\tau)(\gamma+1)} + T^{(\gamma+1)/\epsilon} \|\tilde{w}\|_{X^{s,b}(I)}^{\gamma+1} \\ + (AT^{\frac{1}{2(\gamma+1)}} L^{-(\alpha-\tau)} + T^{1/\epsilon} \|\tilde{w}(t)\|_{X^{s,b}(I)}) (T^{1/\epsilon} \|v(t)\|_{X^{s,b}(I)})^{\gamma}$$

for all $\epsilon > 0$.

Recalling $\tilde{w} \in B_R$ and the hypothesis (57), and observing that the argument given in Appendix A in the case $s = 1$ shows that suitable values of p exist under the hypotheses of the present lemma, we have shown

$$\|\Phi(\tilde{w})\|_{X^{s,b}(I)} \leq C \left(A^{\gamma+1} T^{\frac{1}{2}} L^{-(\alpha-\tau)(\gamma+1)} + T^{(\gamma+1)/\epsilon} R^{\gamma+1} \right. \\ \left. + AT^{\frac{1}{2(\gamma+1)} + \frac{2}{\epsilon}} L^{-(\alpha-\tau)} B^{\gamma} + T^{(\gamma+1)/\epsilon} R B^{\gamma} \right) \\ \leq C \left(\max\{T^{\frac{1}{2}}, T^{\frac{1}{2(\gamma+1)} + \frac{2}{\epsilon}}\} L^{-(\alpha-\tau)} A(A^{\gamma} + B^{\gamma}) \right. \\ \left. + T^{(\gamma+1)/\epsilon} (R^{\gamma} + B^{\gamma}) R \right).$$

Observing that in the above argument we may indeed choose $p = 2$ (since we are in the case $s = 1$), and recalling the bound (47) on ϵ , we may take

$$\frac{1}{\epsilon} = \frac{1}{2(\gamma+1)} - \tilde{\epsilon}$$

for any $\tilde{\epsilon} > 0$ (sufficiently small to ensure $\epsilon > 0$). This gives

$$\|\Phi(\tilde{w})\|_{X^{s,b}(I)} \leq C \left(T^{\frac{1}{2} - \tilde{\epsilon}'} L^{-(\alpha-\tau)} A(A^{\gamma} + B^{\gamma}) \right. \\ \left. + T^{\frac{1}{2} - \tilde{\epsilon}'} (R^{\gamma} + B^{\gamma}) R \right).$$

for any $\tilde{\epsilon}' > 0$ (choose $\tilde{\epsilon} < \tilde{\epsilon}'/(\gamma+1)$). Then, setting

$$\tau = \frac{3}{2} - \frac{1}{\gamma+1} + \tilde{\epsilon}'$$

we obtain

$$\begin{aligned} \|\Phi(\tilde{w})\|_{X^{s,b}(I)} &\leq C \left(T^{\frac{1}{2}-\tilde{\epsilon}'} L^{-\alpha+\frac{3}{2}-\frac{1}{\gamma+1}+\tilde{\epsilon}'} A(A^\gamma + B^\gamma) \right. \\ &\quad \left. + T^{\frac{1}{2}-\tilde{\epsilon}'} (R^\gamma + B^\gamma) R \right). \end{aligned}$$

We now conclude the argument. Setting

$$R = 2CT^{\frac{1}{2}-\tilde{\epsilon}'} L^{-\alpha+\frac{3}{2}-\frac{1}{\gamma+1}+\tilde{\epsilon}'} A(A^\gamma + B^\gamma)$$

and choosing $T > 0$ small enough to ensure that

$$\begin{aligned} C_*(T) &:= CT^{\frac{1}{2}-\tilde{\epsilon}'} (R^\gamma + B^\gamma) \\ &\lesssim T^{\frac{1}{2}-\tilde{\epsilon}'} (A^\gamma (A^{\gamma^2} + B^{\gamma^2}) + B^\gamma) \end{aligned}$$

satisfies

$$C_*(T) < \frac{1}{2},$$

we obtain

$$\|\Phi(\tilde{w})\|_{X^{s,b}(I)} \leq R$$

as desired. Thus, the map $\tilde{w} \mapsto \Phi(\tilde{w})$ carries the set B_R to itself. Analogous arguments now imply that this map is a contraction on B_R . The desired conclusion now follows directly from the contraction mapping theorem, completing the proof of Lemma 6.2. \square

6.2. Proof of Theorem 1.2 – iterative argument. Having established the local well-posedness results of Lemma 6.1 and Lemma 6.2 for the initial-value problems (50) and (51), we now turn to the iteration which leads to the global well-posedness result of Theorem 1.2 for (NLW).

Before proceeding to proof of Theorem 1.2, we shall give a brief schematic of the argument, outlining the choice of initial data for each successive application of Lemma 6.1 and Lemma 6.2, and introducing suitable notation for the construction. We shall then turn to the proof of Theorem 1.2, making use of the notation we have established.

Description of the first iteration

Fix $\epsilon > 0$ and $T > 0$ and let $\delta > 0$ and $L \in \mathbb{N}$ be fixed parameters to be determined in the course of the argument. We begin with the construction of the desired solution u to (NLW) on the initial time interval $[0, \delta]$. For this, we partition the initial data $\phi = \phi_\omega$ in the form

$$\phi = \phi_1 + \phi_2$$

with

$$\phi_1^{(0)}(x) = \sum_{\{(n,n'): z_{n,n'} < L\}} \frac{g_{n,n'}(\omega)}{(z_{n,n'})^\alpha} e_n(x_1, x_2) e^{2\pi i n' x_3}$$

and

$$\phi_2^{(0)}(x) = \sum_{\{(n,n'): z_{n,n'} \geq L\}} \frac{g_{n,n'}(\omega)}{(z_{n,n'})^\alpha} e_n(x_1, x_2) e^{2\pi i n' x_3}$$

for all $x = ((x_1, x_2), x_3) \in B_2 \times \mathbb{T}$.

We now apply Lemma 6.1 and Lemma 6.2 to construct solutions $v^{(0)}$ and $w^{(0)}$ to (50) and (51) (with data $\phi_1^{(0)}$ and $\phi_2^{(0)}$, respectively) on $[0, \delta]$, provided that $\delta > 0$ is chosen sufficiently small and that ω is chosen outside a set of measure $O(\epsilon)$ (the details of this procedure are given in the proof of Theorem 1.2 below). We then set

$$u(t, x) := v^{(0)}(t, x) + w^{(0)}(t, x), \quad t \in [0, \delta], x \in B_2 \times \mathbb{T}. \quad (65)$$

Description of the subsequent iterations

Let ϵ , T , δ and L be as above. Suppose that we have constructed a solution u up to time $k\delta$ for some $k \geq 1$. We now outline the extension of this construction to the interval $[0, (k+1)\delta]$. The argument is again based on the local well-posedness results of Lemma 6.1 and 6.2. In this case, the data will be chosen as

$$\phi_1^{(k)}(x) := u(k\delta, x) - S(k\delta)\phi_2^{(0)}(x)$$

and

$$\phi_2^{(k)}(x) := S(k\delta)\phi_2^{(0)}(x)$$

for $x \in B_2 \times \mathbb{T}$.

Applications of Lemma 6.1 and Lemma 6.2 now give solutions $v^{(k)}$ and $w^{(k)}$ to (50) (with data $\phi_1^{(k)}$) and (51) (with data $\phi_2^{(k)}$ and $v = v^{(k)}$) on the interval $[k\delta, (k+1)\delta]$, provided that δ is chosen sufficiently small and that ω remains outside a set of measure $O(\epsilon)$.

With $v^{(k)}$ and $w^{(k)}$ in hand, we extend the definition of u to the interval $[0, (k+1)\delta]$ by writing

$$u(t, x) = v^{(k)}(t, x) + w^{(k)}(t, x), \quad t \in [k\delta, (k+1)\delta], x \in B_2 \times \mathbb{T}.$$

We are now ready to give the proof of Theorem 1.2. The structure of the argument follows the outline we have just described, keeping careful track of the local existence time guaranteed by each application of Lemma 6.1 and Lemma 6.2 (note that these existence times depend strongly on the norm of the constructed solution at each iteration).

Proof of Theorem 1.2. Let $\epsilon > 0$ and $T > 0$ be given. Fix $\delta > 0$ and $L \in \mathbb{N}$ to be chosen later in the argument, and let $\phi_1^{(0)}$ and $\phi_2^{(0)}$ be as defined in Section 6.2.1 above. Now, fix $\sigma < \alpha - 1$ and $M \geq 1$, and note that in view of the frequency support of $\phi_1^{(0)}$ and Lemma 3.1 we have

$$\|\phi_1^{(0)}\|_{H_x^1} \lesssim L^{1-\sigma} \|\phi_1^{(0)}\|_{H_x^\sigma} \lesssim L^{1-\sigma} M$$

and

$$\|\phi_1^{(0)}\|_{L_x^{\gamma+2}} \lesssim M^{\frac{2}{\gamma+2}} \leq (L^{1-\sigma} M)^{\frac{2}{\gamma+2}}$$

for all ω outside a set of measure $O(\exp(-cM^{\frac{4}{\gamma+2}}))$. It now follows from Lemma 6.1 that for some $\delta > 0$ satisfying

$$\delta \leq c_1(L^{1-\sigma}M)^{c_2},$$

where $c_1 = c_1(\gamma), c_2 = c_2(\gamma) > 0$ are constants, there exists a solution $v^{(0)}$ on the interval $[0, \delta]$ to the initial value problem (50) with data $\phi_1^{(0)}$, with the bounds

$$\|v^{(0)}\|_{X^{1,b}(I)} \leq CL^{1-\sigma}M$$

for suitable $b > \frac{1}{2}$, and

$$\|v^{(0)}(\delta)\|_{H_x^1} \leq C(L^{1-\sigma}M).$$

We now apply Lemma 6.2 to obtain that if δ satisfies

$$\delta \leq c(M + CL^{1-\sigma}M)^{-c'} \leq c(L^{1-\sigma}M)^{-c'}$$

and ω lies outside a set of measure $O(\exp(-cM^2))$, then there also exists a solution $w^{(0)}$ defined on $[0, \delta]$ to the initial value problem (51) with $v = v^{(0)}$ and initial data $\phi_2 = \phi_2^{(0)}$. Moreover, we obtain the bound

$$\|w^{(0)} - S(t)\phi_2^{(0)}\|_{X^{1,b}([0,\delta])} \leq C\delta^{\frac{1}{2}-\tilde{c}'}L^{-\alpha+\frac{3}{2}-\frac{1}{\gamma+1}+\tilde{c}'}M^{\gamma+1}L^{(1-\sigma)\gamma}$$

by virtue of the estimate $\|w^{(0)} - S(t)\phi_2\|_{X^{1,b}} \leq R$ arising from the construction, with R as in the proof of Lemma 6.2.

Define $u : [0, \delta] \times B_2 \times \mathbb{T} \rightarrow \mathbb{C}$ by (65). To conclude the proof, we must now show that u can be extended to the full interval $[0, T]$ as a solution of (5). We proceed by iteratively constructing the solution on intervals of the form $[k\delta, (k+1)\delta]$ for $k = 1, 2, \dots, \lfloor \frac{T}{\delta} \rfloor + 1$.

Suppose that for some $1 \leq k \leq \lfloor \frac{T}{\delta} \rfloor + 1$ we have constructed the solution on $[0, k\delta]$ with bounds

$$E(\phi_1^{(k'-1)}) \leq C_0(L^{1-\sigma}M)^2, \quad (66)$$

$$\|v^{(k')}\|_{X^{1,b}([(k'-1)\delta, k'\delta])} \leq C_0L^{1-\sigma}M \quad (67)$$

and

$$\begin{aligned} \|w^{(k')} - S(t - (k' - 1)\delta)\phi_2^{(k'-1)}\|_{X^{1,b}([(k'-1)\delta, k'\delta])} \\ \leq C_0\delta^{\frac{1}{2}-\tilde{c}'}L^{-\alpha+\frac{3}{2}-\frac{1}{\gamma+1}+\tilde{c}'}M^{\gamma+1}L^{(1-\sigma)\gamma} \end{aligned} \quad (68)$$

holding for fixed $\tilde{c}' > 0$ and all $k' = 1, \dots, k$, with $C_0 > 0$ sufficiently large. We now extend the construction to $[0, (k+1)\delta]$, preserving C_0 . Let $\phi_1^{(k)}$ and $\phi_2^{(k)}$ be given as in Section 6.2.2. Note that by the construction, we have

$$\begin{aligned} E(\phi_1^{(k)}) &= C \left(E(\phi_1^{(0)}) + \sum_{k'=1}^k [E(\phi_1^{(k')}) - E(\phi_1^{(k'-1)})] \right) \\ &\leq C \left((L^{1-\sigma}M)^2 + \sum_{k'=1}^k E_{k'} \right) \end{aligned} \quad (69)$$

with

$$E_{k'} := E(v^{(k'-1)}(k'\delta) + w^{(k'-1)}(k'\delta) - S(k'\delta)\phi_2^{(0)}) - E(v^{(k'-1)}(k'\delta))$$

for $k' = 1, \dots, k$, where in passing to the last line in (69) we have used the energy conservation

$$E\left(\phi_1^{(k'-1)}\right) = E\left(v^{(k'-1)}(k'\delta)\right)$$

satisfied by each $v^{(k'-1)}$ as a solution of (50).

We now estimate the energy increments $E_{k'}$. Let $k' \in \mathbb{N}$ be given with $1 \leq k' \leq k$, and write

$$E_{k'} = \frac{1}{2}(I) + \frac{1}{\gamma+2}(II) \quad (70)$$

by setting

$$(I) := \|v^{(k'-1)}(k'\delta) + w^{(k'-1)}(k'\delta) - S(k'\delta)\phi_2^{(0)}\|_{H_x^1}^2 - \|v^{(k'-1)}(k'\delta)\|_{H_x^1}^2$$

and

$$(II) := \|v^{(k'-1)}(k'\delta) + w^{(k'-1)}(k'\delta) - S(k'\delta)\phi_2^{(0)}\|_{L_x^{\gamma+2}}^{\gamma+2} - \|v^{(k'-1)}(k'\delta)\|_{L_x^{\gamma+2}}^{\gamma+2}.$$

To bound (I), we note that the elementary inequality $|a+b|^2 - |b|^2 \lesssim |a|(|a| + |b|)$ (c.f. (48)) implies

$$(I) \lesssim \int_{B_2 \times \mathbb{T}} \left| \nabla \left(w^{(k'-1)}(k'\delta) - S(k'\delta)\phi_2^{(0)} \right) \right| \cdot \left(\left| \nabla v^{(k'-1)}(k'\delta) \right| + \left| \nabla \left(w^{(k'-1)}(k'\delta) - S(k'\delta)\phi_2^{(0)} \right) \right| \right) dx.$$

Using the Hölder inequality combined with the bounds (67), (68) and the embedding $X^{1,b} \hookrightarrow L_t^\infty H_x^1$ given by (16), the quantity (I) is therefore bounded by a multiple of

$$W L^{1-\sigma} M + W^2 \quad (71)$$

with

$$W := \delta^{\frac{1}{2}-\tilde{\epsilon}'} L^{(1-\sigma)\gamma-\alpha+\frac{3}{2}-\frac{1}{\gamma+1}+\tilde{\epsilon}'} M^{\gamma+1}. \quad (72)$$

We now bound (II) by proceeding in a similar way, obtaining

$$\begin{aligned} (II) &\lesssim \int_{B_2 \times \mathbb{T}} \left| w^{(k'-1)}(k'\delta) - S(k'\delta)\phi_2^{(0)} \right| \\ &\quad \cdot \left(\left| v^{(k'-1)}(k'\delta) \right|^{\gamma+1} + \left| w^{(k'-1)}(k'\delta) - S(k'\delta)\phi_2^{(0)} \right|^{\gamma+1} \right) dx \\ &\leq \|w^{(k'-1)}(k'\delta) - S(k'\delta)\phi_2^{(0)}\|_{L_x^{\gamma+2}}^{\gamma+1} \|v^{(k'-1)}(k'\delta)\|_{L_x^{\gamma+2}}^{\gamma+1} \\ &\quad + \|w^{(k'-1)}(k'\delta) - S(k'\delta)\phi_2^{(0)}\|_{L_x^{\gamma+2}}^{\gamma+2} \\ &\lesssim \|w^{(k'-1)}(t) - S(t)\phi_2^{(0)}\|_{L_t^\infty H_x^1} E(\phi_1^{(k'-1)})^{\frac{\gamma+1}{\gamma+2}} \\ &\quad + \|w^{(k'-1)}(k'\delta) - S(k'\delta)\phi_2^{(0)}\|_{L_t^\infty H_x^1}^{\gamma+2} \end{aligned}$$

where each of the L_t^∞ norms on the last line are taken over the set $[(k'-1)\delta, k'\delta]$; note that the last line here follows from the Sobolev embedding $\|f\|_{L_x^{\gamma+2}} \lesssim \|f\|_{H_x^{\frac{3\gamma}{2(\gamma+2)}}}$

and the fact that $\gamma < 4$ implies $\frac{3\gamma}{2(\gamma+2)} < 1$. Recalling the bounds (66)–(68) and

using the embedding (16) as above, we obtain that (II) is bounded by a multiple of

$$W(L^{1-\sigma}M)^{\frac{2(\gamma+1)}{\gamma+2}} + W^{\gamma+2}. \quad (73)$$

Assembling (70) with the bounds (71) and (73) for (I) and (II) and the definition (72) of W , we obtain

$$(69) \leq C \left((L^{1-\sigma}M)^2 + \sum_{k'=1}^{\lfloor \frac{T}{\delta} \rfloor + 1} \left[(L^{1-\sigma}M + W + (L^{1-\sigma}M)^{\frac{2(\gamma+1)}{\gamma+2}} + W^{\gamma+1})W \right] \right).$$

It then follows that we have

$$E(\phi_1^{(k)}) \leq C \left((L^{1-\sigma}M)^2 + \frac{T}{\delta} \left[\delta^{\frac{1}{2}-\tilde{c}'} L^\eta M^{(\gamma+1)(\gamma+2)} \right] \right),$$

where we have chosen

$$\eta > (1-\sigma) \left(\gamma + \frac{2(\gamma+1)}{\gamma+2} \right) - \alpha + \frac{3}{2} - \frac{1}{\gamma+1}.$$

In order to ensure

$$E(\phi_1^{(k)}) \leq C(L^{1-\sigma}M)^2$$

and complete the inductive step (since once this inequality is in hand, we immediately obtain $v^{(k)}$ and $w^{(k)}$ solving (50) and (51) and satisfying the bounds (66)–(68); indeed, this follows from an application of Lemma 6.1 and Lemma 6.2 identical to the construction of $v^{(0)}$ and $w^{(0)}$), we therefore need to choose δ , L to satisfy

$$CT\delta^{\frac{1}{2}-\tilde{c}'} L^\eta M^{(\gamma+1)(\gamma+2)} \leq L^{2(1-\sigma)} M^2. \quad (74)$$

Recall that $\delta \sim (L^{1-\sigma}M)^{-c'}$ in our setting. By examining the proofs of Lemma 6.1 and Lemma 6.2 we note that we can choose $p = 2$ in these arguments, and it therefore suffices to work with

$$c' > 2\gamma$$

It is then possible to satisfy inequality (74) (by choosing L sufficiently large) whenever σ satisfies

$$(1-\sigma) \left(2\gamma + \frac{2(\gamma+1)}{\gamma+2} - 2 \right) - \alpha + \frac{3}{2} - \frac{1}{\gamma+1} < 0$$

Since σ was an arbitrary value in the range $0 < \sigma < \alpha - 1$, this condition is equivalent to

$$(2-\alpha) \left(2\gamma + \frac{2(\gamma+1)}{\gamma+2} - 2 \right) - \alpha + \frac{3}{2} - \frac{1}{\gamma+1} < 0,$$

which is precisely the condition of Theorem 1.2. This completes the proof of the theorem. \square

APPENDIX A. THE CHOICE OF s AND p IN THE PROOF OF THEOREM 1.1 (AND
SUBSEQUENT RESULTS)

In this appendix, we summarize the derivation of the condition (10) on the strength γ of the nonlinearity in the local well-posedness result, Theorem 1.1. Recall that the initial value problem (NLW) under consideration in this paper is equipped with nonlinearity $|u|^\gamma u$; for the reformulated equation (5), the nonlinearity takes the form $F(u) = |\operatorname{Re}(u)|^\gamma \operatorname{Re}(u)$.

In particular, we show how the condition (10) allows for the choice of the parameters $s > 0$ and $p \geq 2$ used in the proof of Theorem 1.1. We shall choose such values to satisfy the condition of the nonlinear $X^{s,b}$ estimate in Lemma 4.1 together with the conditions (42) and (45), which correspond respectively to applications of the probabilistic estimate of Lemma 3.1 and the $X^{s,b} \hookrightarrow L_x^p L_t^q$ embedding of Lemma 4.2.

In what follows, we will first choose a value for s , and then make a suitable choice of $p \geq 2$ to ensure that the desired conditions are satisfied. We shall consider three cases depending on the value of s to be chosen. The derivation of condition (10) corresponds to Case 2, while Case 1 gives suitable choices of s and p for the global theory — in particular, the deterministic and perturbed probabilistic local well-posedness results of Lemma 6.1 and Lemma 6.2, which are used to prove the global well-posedness result of Theorem 1.2. The remaining case, Case 3 gives an extended range for γ when s is allowed to be in the range $0 < s < 1/2$.

Case 1: $s = 1$.

We remark that this is also the relevant case for the global result Theorem 1.2 — recall that this theorem is proved in Section 6. In particular, the same considerations we discuss here also apply in the case of the proofs of Lemma 6.1 and Lemma 6.2.

In this case, the condition of Lemma 4.1 reads

$$p \geq 2, \tag{75}$$

while (42) and (45) take the forms

$$2 \leq p(\gamma + 1) < \rho_*(\alpha) \tag{76}$$

and

$$1 > \frac{1}{2} - \frac{1}{2(\gamma + 1)} + \max \left\{ \frac{1}{2} - \frac{2}{p(\gamma + 1)}, 0 \right\}. \tag{77}$$

Note that the first inequality in (76) is satisfied as a consequence of $p \geq 2$ and $\gamma \geq 1$. Moreover, without any loss of generality we may suppose further that $p(\gamma + 1) \geq 4$, so that the maximum in (77) is equal to $\frac{1}{2} - \frac{2}{p(\gamma + 1)}$. Indeed, we will take p close to the upper bound in (76) (when it is finite); this value is larger than 4 for all α in the relevant range $\alpha > 1$. It then follows that (77) is automatically satisfied for all p .

It therefore remains to choose $p \geq \max\{2, \frac{4}{\gamma + 1}\}$ such that the upper bound in (76) is satisfied. For this, we consider two subcases, $1 < \alpha < \frac{3}{2}$ and $\alpha \geq \frac{3}{2}$.

When $1 < \alpha < \frac{3}{2}$ (corresponding to the values of α for which $\rho_*(\alpha) < \infty$), the restriction on p becomes

$$\max\left\{2, \frac{4}{\gamma+1}\right\} \leq p < \frac{4}{(3-2\alpha)(\gamma+1)}.$$

It then follows that choice of p is possible whenever γ satisfies

$$1 \leq \gamma \leq G_1 := \frac{2\alpha-1}{3-2\alpha}. \quad (78)$$

We now consider the subcase $\alpha \geq \frac{3}{2}$ (when $\rho_*(\alpha) = \infty$). In this case, the only restriction becomes $p \geq \max\{2, \frac{4}{\gamma+1}\}$, so that choosing p as this threshold value works for all

$$1 \leq \gamma < \infty.$$

Case 2: $\frac{1}{2} \leq s < 1$.

In this case, again supposing $p(\gamma+1) \geq 4$, the condition of Lemma 4.1, (42), and (45) read

$$p > \frac{6}{5-2s} \quad (79)$$

$$2 \leq p(\gamma+1) < \rho_*(\alpha) \quad (80)$$

$$s > 1 - \frac{1}{2(\gamma+1)} - \frac{2}{p(\gamma+1)}. \quad (81)$$

Writing (79) as

$$s < \frac{5}{2} - \frac{3}{p},$$

we see that, by (81), s must lie in the range

$$1 - \frac{1}{2(\gamma+1)} - \frac{2}{p(\gamma+1)} < s < \frac{5}{2} - \frac{3}{p}. \quad (82)$$

It therefore suffices to choose $s \in [1/2, 1)$ and $p \geq \max\{2, \frac{4}{\gamma+1}\}$ satisfying (80) and (82). As in our analysis of Case 1 above, we consider two subcases, depending on the value of α .

In the first subcase, when $1 < \alpha < \frac{3}{2}$, we take

$$p = \frac{4}{(3-2\alpha)(\gamma+1)} - \epsilon$$

and observe that choice of s satisfying (82) is possible when

$$1 - \frac{1}{\gamma+1} \left(\frac{1}{2} + \frac{(3-2\alpha)(\gamma+1)}{2} + \tilde{\epsilon} \right) < \frac{5}{2} - \frac{3(3-2\alpha)(\gamma+1)}{4} - \tilde{\epsilon},$$

where $\tilde{\epsilon} > 0$ is chosen to satisfy

$$\frac{2}{\left(\frac{4}{(3-2\alpha)(\gamma+1)} - \epsilon\right)} = \frac{(3-2\alpha)(\gamma+1)}{2} + \tilde{\epsilon}.$$

This implies that choice of s is possible (for $\tilde{\epsilon}$ sufficiently small) when

$$\max \left\{ 1 - \frac{1}{\gamma+1} \left(\frac{1}{2} + \frac{(3-2\alpha)(\gamma+1)}{2} \right), \frac{1}{2} \right\} < \min \left\{ \frac{5}{2} - \frac{3(3-2\alpha)(\gamma+1)}{4}, 1 \right\} \quad (83)$$

is satisfied (indeed, we have recalled the condition $\frac{1}{2} \leq s < 1$ and observed that, for $A < B$ and $c < d$, one has $A - \epsilon c < B - \epsilon d$ provided $0 < \epsilon < \frac{B-A}{d-c}$).

The condition (83) is equivalent to the two conditions

$$\gamma < \frac{6\alpha - 1}{3(3-2\alpha)}$$

and

$$(\gamma+1) - \left(\frac{1}{2} + \frac{(3-2\alpha)(\gamma+1)}{2} \right) < \frac{5}{2}(\gamma+1) - \frac{3(3-2\alpha)(\gamma+1)^2}{4}.$$

We now observe that the second of these conditions can be written as

$$-\left(\frac{3(3-2\alpha)}{4} \right)(\gamma+1)^2 + \left(\frac{3}{2} + \frac{3-2\alpha}{2} \right)(\gamma+1) + \frac{1}{2} > 0,$$

and this inequality holds for γ in the range

$$\frac{6-2\alpha-\sqrt{4\alpha^2-36\alpha+54}}{3(3-2\alpha)} < \gamma+1 < \frac{6-2\alpha+\sqrt{4\alpha^2-36\alpha+54}}{3(3-2\alpha)}. \quad (84)$$

Noting that for $\alpha \in (1, \frac{3}{2})$, the lower bound in (84) is negative, we obtain that choice of s and p is possible whenever γ satisfies

$$1 \leq \gamma < G_2 := \min \left\{ \frac{6\alpha-1}{3(3-2\alpha)}, \frac{4\alpha-3+\sqrt{4\alpha^2-35\alpha+54}}{3(3-2\alpha)} \right\}. \quad (85)$$

This is precisely the restriction on γ appearing in the statement of Theorem 1.1 for $\alpha \in (1, \frac{3}{2})$.

Turning to the subcase $\alpha \geq \frac{3}{2}$, it suffices to observe that the conditions (80) and (82) are satisfied for all pairs (s, p) satisfying $\frac{1}{2} \leq s < 1$, $p \geq \max\{2, \frac{4}{\gamma+1}\}$ and

$$1 - \frac{1}{2(\gamma+1)} - \frac{2}{p(\gamma+1)} < s < \frac{5}{2} - \frac{3}{p}.$$

It follows that choice of s is then possible whenever p is chosen large enough.

Case 3: $0 < s < \frac{1}{2}$.

For this case, we argue in a similar way as in Case 2. Assuming again that p will satisfy $p(\gamma+1) \geq 4$, the relevant conditions become

$$p > \frac{2}{2-s} \quad (86)$$

together with (80) and (81).

Since for $\alpha \geq \frac{3}{2}$, the previous cases showed that we could choose (s, p) as needed for all $p \geq 1$, in this case it suffices to consider only the situation $1 < \alpha < \frac{3}{2}$. The analogue of (83) is then

$$1 - \frac{1}{\gamma + 1} \left(\frac{1}{2} + \frac{(3 - 2\alpha)(\gamma + 1)}{2} \right) < \min \left\{ 2 - \frac{(3 - 2\alpha)(\gamma + 1)}{2}, \frac{1}{2} \right\}$$

which is equivalent to

$$\gamma < \frac{3 - 2\alpha}{2(\alpha - 1)}$$

and

$$-\left(\frac{3 - 2\alpha}{2}\right)(\gamma + 1)^2 + \left(1 + \frac{3 - 2\alpha}{2}\right)(\gamma + 1) + \frac{1}{2} > 0. \quad (87)$$

The inequality (87) holds for γ in the range

$$\frac{5 - 2\alpha - \sqrt{4\alpha^2 - 28\alpha + 37}}{2(3 - 2\alpha)} < \gamma + 1 < \frac{5 - 2\alpha + \sqrt{4\alpha^2 - 28\alpha + 37}}{2(3 - 2\alpha)}$$

We therefore obtain that choice of s, p is possible whenever γ satisfies

$$1 \leq \gamma < G_3 := \min \left\{ \frac{3 - 2\alpha}{2(\alpha - 1)}, \frac{2\alpha - 1 + \sqrt{4\alpha^2 - 28\alpha + 37}}{2(3 - 2\alpha)} \right\}. \quad (88)$$

This concludes our analysis of the case $0 < s < \frac{1}{2}$.

REFERENCES

- [1] J. Bourgain. Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.* 166 (1994), 1–24.
- [2] J. Bourgain. Invariant measures for the 2D-Defocusing Nonlinear Schrödinger Equation. *Comm. Math. Phys.* 176 (1996), 421–445.
- [3] J. Bourgain. Invariant measures for the Gross-Piatevskii equation. *J. Math. Pures Appl.* (9) 76 (1997), no. 8, 649–702.
- [4] J. Bourgain. Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity. *Int. Math. Res. Not.* 5 (1998), 253–283.
- [5] J. Bourgain and A. Bulut. Gibbs measure evolution in radial nonlinear wave and Schrödinger equations on the ball. *C. R. Mathématique* 350 (2012) 11–12, p. 571–575.
- [6] J. Bourgain and A. Bulut. Almost sure global well posedness for the radial nonlinear Schrödinger equation on the unit ball I: the 2D case. To appear in *Ann. Inst. H. Poincaré* (2013).
- [7] J. Bourgain and A. Bulut. Almost sure global well posedness for the radial nonlinear Schrödinger equation on the unit ball II: the 3D case. *J. Eur. Math. Soc.* 16 (2014), no. 6, 1289–1325.
- [8] J. Bourgain and A. Bulut. Invariant Gibbs measure evolution for the radial nonlinear wave equation on the 3d ball. *J. Funct. Anal.* 266 (2014), no. 4, 2319–2340.
- [9] N. Burq. Random data Cauchy theory for dispersive partial differential equations. *Proceedings of the International Congress of Mathematicians. Volume III*, 1862–1883. Hindustan Book Agency, New Delhi, 2010.
- [10] N. Burq, G. Lebeau and F. Planchon. Global existence for energy critical waves in 3-D domains. *J. Amer. Math. Soc.* 21 (2008), no. 3, 831–845.
- [11] N. Burq and N. Tzvektov. Random data Cauchy theory for supercritical wave equations. I. Local theory. *Invent. Math.* 173 (2008), no. 3, 449–475.
- [12] N. Burq and N. Tzvektov. Random data Cauchy theory for supercritical wave equations. II. A global existence result. *Invent. Math.* 173 (2008), no. 3, 447–496.

- [13] N. Burq and N. Tzvetkov. Probabilistic well-posedness for the cubic wave equation. (2011) J. Eur. Math. Soc. 16 (2014), no. 1, 1–30.
- [14] J. Colliander and T. Oh. Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$. Duke Math. J. 161 (2012), no. 3, 367–544.
- [15] C. Kenig, G. Ponce and L. Vega. Global well-posedness for semilinear wave equations. Comm. Par. Diff. Eq. 25 (2000), 1741–1752.
- [16] J. Lebowitz, H. Rose and E. Speer. Statistical mechanics of the nonlinear Schrödinger equation. J. Statis. Phys. 50 (1988), no. 3–4, 657–687.
- [17] J. Luhrmann and D. Mendelson. Random data Cauchy theory for nonlinear wave equations of power-type on \mathbb{R}^3 . Preprint (2013). arXiv:1309.1225.
- [18] A. Nahmod, T. Oh, L. Rey-Bellet and G. Staffilani. Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS. J. Eur. Math. Soc. 14 (2012), no. 4, 1275–1330.
- [19] A. Nahmod, N. Pavlovic and G. Staffilani. Almost sure existence of global weak solutions for supercritical Navier-Stokes equations. SIAM J. Math. Anal. 45 (2013), no. 6, 3431–3452.
- [20] A. Nahmod and G. Staffilani. Almost sure well-posedness for the periodic 3D quintic nonlinear Schrödinger equation below the energy space. Preprint (2013), arXiv:1308.1169.
- [21] J. Shatah and M. Struwe. Geometric wave equations. Courant Lecture Notes in Mathematics, 2. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI. 1998.
- [22] N. Tzvetkov. Invariant measures for the nonlinear Schrödinger equation on the disc. Dyn. Partial Diff. Eq. 3 (2006), no. 2, 111–160.
- [23] N. Tzvetkov. Invariant measures for the defocusing nonlinear Schrödinger equation. Ann. Inst. Fourier 58 (2008), no. 7, 2543–2604.

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